Hat Calculus

Hats as Growth Rates

Hats denote growth rates or percentage changes of the underlying level variable. If GDP is 100 in 2000 and grows to 106 in 2001, the total percentage change (106-100)/100 = 0.06 or 6 percent. It is correct to say that the growth rate of GDP was 6 percent during this period. Hats are then made up of "dots" and levels, $\dot{x} = \Delta x/\Delta t$ where x is the level of any variable and t is time. We have

$$\hat{x} = \dot{x}/x$$

Hats are related to logarithms and they follow the same rules.[1]

Rules

Rule 1 Multiplication. Let

$$x = yz$$

where x, y and z are the *levels* of any three variables. The first rule says

$$\hat{x} = \hat{y} + \hat{z}$$

So if the levels are multiplied, then the hats are added.

Rule 2 Division. Let

$$x = y/z$$

The second rule says

$$\hat{x} = \hat{y} - \hat{z}$$

So if the levels are divided, then the hats are subtracted.

Rule 3 Multiplication by a constant. We have

$$x = ay$$

with a constant. The third rule says

$$\hat{x} = \hat{y}$$

So if the level is multiplied by a constant, it drops out. *Exercise: prove that this rule is true using Rule 1.* Remark: since a could equal $1/b$ the same is true for division by a constant.

Rule 4 Exponents. We have

$$x = y^a$$

be the *levels* of the two variable x and y with a constant. The fourth rule says

$$\hat{x} = a\hat{y}$$

So if the levels are raised to a constant exponent then, in this case, the constant does not disappear.

Rule 5 Now there is a special case of the exponent rule when $y = e$, the base of the natural logarithm system:

$$y = e^{gt}$$

In this case, we have $\hat{y} = g$.

Rule 6 Weighted average. Say we have

$$x = ay + bz$$

with a and b constant. Calculate the rate of growth of x is more complicated and requires that we write

$$\hat{x} = \frac{ay}{ay + bz}\hat{y} + \frac{bz}{ay + bz}\hat{z}$$

[1] It not necessary to know this to understand the rest of the handout, but if we have $y = ln(x)$ then the derivative of y is $\frac{dy}{dt} = \frac{dx/dt}{x} = \hat{x}$. So hats are the change in logs.

So when a variable is defined as the weighted sum of the two levels y and z with weights a and b, then the growth rate of x is the weighted sum of the growth rates of y and z. This rule is far less useful since it requires that the levels be involved in the calculation of the growth rate of x. Since the levels of y and z are changing, this rule is often difficult to apply.

Rule 7 Average rate of growth. Let

$$x_t = (1+g)^t x_0$$

where x is any variable and x_0 is the initial value and the final value is x_t. We can calculate the average growth rate, g, by solving this equation

$$g = (\frac{x_t}{x_0})^{1/t} - 1$$

Note that like all applications of calculus, these rules only apply for small changes; for large changes they are only approximations.

Examples

1. Nominal GDP grows at 6% but inflation is 4%. What is the approximate growth rate of real GDP? *Answer: Apply rule 1 to* $Y = pX\hat{X} = 2\%$

2. Output per worker is defined as

$$\rho = X/L$$

where X is the real GDP and L is number of employed workers. We know that productivity usually grows at around 1%. If employment is up by 2% this year, what is the growth rate of real GDP? *Answer: apply rule 2*

$$\begin{aligned} \hat{\rho} &= 1\% \\ \hat{L} &= 2\% \\ \hat{X} &= 3\% \end{aligned}$$

3. The rate of inflation is $\hat{p}$ and it is equal to 5% per year. What rate of growth in nominal GDP, Y, would be require for real growth to be 3%? *Answer: apply rule 1*

$$\begin{aligned} Y &= pX \\ \hat{Y} &= \hat{p} + \hat{X} \\ 8\% &= 5\% + 3\% \end{aligned}$$

4. With the same rate of inflation as in the previous problem what rate of growth of *nominal* wages is required to keep real wages constant? *Answer: apply rule 2*

$$\begin{aligned} \widehat{w/p} &= \hat{w} - \hat{p} \\ 0 &= \hat{w} - 5\% \\ \hat{w} &= 5\% \end{aligned}$$

5. The share of labor, σ is defined as

$$\sigma = \frac{wL}{pX}$$

where L = employment; w = wage rate $w_r = w/p$ = real wage and $\rho = X/L$ = productivity. If $\hat{\sigma} = 0$, what is the relationship between the real wage and productivity growth?

Answer: apply rule 2 to get $\hat{w_r} = \hat{\rho}$. The real wage must grow at the same rate as productivity.

6. Consider the production function $y = ak$ where y = output per person, a is a constant and k is the capital labor ratio. If the capital-labor ratio is growing at 2% per year, what is the rate of growth of output per person?

   Answer: Apply rule 3. Since a is constant, output per person also grows at 2% per year.

7. If the rate of growth of the share of labor is zero and the rate of employment growth is 2%, what is the rate of growth of the real wage required for 4% growth rate of real GDP?

   Answer: apply rule 2 to get

$$\begin{aligned} 0 &= \hat{w} + \hat{L} - \hat{p} - \hat{X} \\ &= \hat{w} - \hat{p} + 2 - 4 \\ \hat{w}_r &= \hat{w} - \hat{p} = 2\% \end{aligned}$$

   So the answer is that the real wage w_r grows by 2%.

8. If the rate of inflation is 8% in the previous problem what must be the rate of growth of nominal wages?

   Answer: apply rule 2 to get

$$\hat{w}_r = \hat{w} - 8 = 2\%$$

   Answer: $\hat{w}$ =10%.

9. Define the rate of unemployment as

$$u = \frac{\bar{L} - L}{\bar{L}}$$

   where $\bar{L}$ is the *labor force* and L is the current level of employment.

10. Now write the number of unemployed as $U = \bar{L} - L$ and then u can be expressed

$$u = \frac{U}{\bar{L}}$$

   Can we say that $\hat{u} = \hat{U} - \hat{\bar{L}}$? *Answer: yes, this is an application of rule 2.*

11. If $\hat{\bar{L}} = 0.1\%$, a 2% change in the number of unemployed gives what percentage in the unemployment rate? *Answer: apply rule 2* $\hat{u} = 2\% - 1\% = 1\%$. But this means that if the rate of unemployment were 3%, the change in the unemployment rate would be 1% of 3%. It would then change to 3.03, an insignificant change. This illustrates a basic problem with hats. If u is *already a percentage* so that $\hat{u}$ is a percentage of a percentage. This is not commonly used in economics (or anywhere else for that matter.) See how to deal with this problem in the section on Okun's law below.

12. Let the growth in employment be zero. If the economy achieves a real growth rate of 3% with 5% inflation and nominal wages increase by 4% what happens to the share of labor?

   Answer: apply rules 1 and 2

$$\begin{aligned} \hat{\sigma} &= \hat{w} + \hat{L} - \hat{p} - \hat{X} \\ &= 4\% + 0\% - 5\% - 3\% \\ &= -4\% \end{aligned}$$

   Again, remember that the share of labor is itself a percentage.

13. Let capacity output be Q and define *capacity utilization* as

$$\mu = X/Q$$

   Say μ is 0.85. If Q grows by 3.5% per year, but output grows by 3% what happens to capacity utilization? *Answer: it falls by 0.5%.*

14. Let there be two sectors fish and cocos. Fish output grows at 4% per year and cocos grows at 1% per year. If GDP consists of 20 fish and 40 cocos, what is the rate of growth of GDP? *Answer: apply rule 6.* GDP in terms of cocos is $Y = p_f 20 + 40$, where p_f is the opportunity cost of fish in terms of cocos and say it is 3. GDP is then 100 and the rate of growth of GDP is then

$$\hat{X} = 3(20)/100(4) + 40/100(1) = 2.8$$

. where the weights are 3(40)/100= 0.6 and 0.4.

15. When compounding interest is instantaneous, the equation for the value of an asset is

$$V = e^{gt}$$

where g is the instantaneous rate of growth is 2.3%. See (The Story of e). What is the rate of growth the $\hat{V}$?

Answer: apply rule 5. Since the g is the instantaneous rate of growth, $\hat{V} = 2.3$.

16. Fill in the following table: X is real GDP and Y is nominal GDP, $\hat{p}$ is the rate of inflation.

time	X	Y	$\hat{p}$	*Answer*	X	Y	$\hat{p}$
1	100	100	.03	1	100	100	.03
2	101		.02	2	101	103	.02
3	104	110		3	104	110	.04
4		108	.05	4	97	108	.05
5	112	115		5	112	115	−.09

17. What is the *average* rate of growth of real GDP.

Answer: Apply rule 7: The average rate of growth of GDP is the g that gives:

$$100(1+g)^4 = 112$$

Solving this expression for g we have:

$$g = (\frac{112}{100})^{\frac{1}{4}} - 1 = .0287$$

18. Why can we not take the yearly rates of growth of real GDP and average them? *Answer: because of compounding. Each year's average growth rate applies to the previous year's level which includes the growth from all years before. The average is* .02882. This is very close, but larger than the true average.

19. Fill in the following table assuming a rate of growth of the labor force of 1% and a *constant* rate of unemployment.

time	X	L	$\hat{\rho}$	*Answer*	X	L	$\hat{\rho}$
1	100	100	.03	1	100	100.0	.03
2	101			2	101	101.0	.00
3	104			3	104	102.0	.020
4	107			4	107	103.0	.019
5	112			5	112	104.1	.036

20. Consider the production function $y = k^a$ where y = output per person, $a = 0.45$ is a constant and k is the capital labor ratio. If the capital-labor ratio is growing at 2% per year, what is the rate of growth of output per person?

Answer: Apply rule 4. Since a is constant, output per person also grows at (0.45)2% = 0.9% per year.

Okun's Law

Okun's Law says that a 1% increase in the unemployment *rate* cost the economy an approximately 2% in real growth. To use hat calculus with Okun's law, we have to note that

$$\begin{aligned}\hat{u} &= \frac{\Delta u}{u} \\ \Delta u &= \hat{u}u\end{aligned}$$

where Δu is the percentage-point change in the unemployment. For example if the unemployment rate rises by from 3% to 4.5%, then $\Delta u = 1.5$.

21. If there were a 1% increase in the rate of unemployment and if the labor force increased by 0.5%, what is the percentage growth rate in the number of unemployed.

    Answer: From the equation $\hat{u} = \hat{U} - \hat{\bar{L}}$ we would have

    $$\frac{0.01}{u} = \hat{U} - 0.5$$

    since the question says $\Delta u = 0.01$ *not* $\Delta u/u = 0.01$. Hence we must know what the rate of unemployment is to use the hat expression. Let us say that it 4%. We then have:

    $$\frac{0.01}{.04} = \hat{U} - 0.5$$

    and the growth rate of the unemployed is 0.75%.

22. If the number of unemployed is 6 million, what is the difference between hiring and firing for the year?

    Answer $(0.0075)6,000,000 = 45,000$ people.

23. If Okun's law applies, what is the growth rate of real output when the unemployment rate increases from 3.4% to 4.6%?

    Answer:

    $$\begin{aligned}4.6\% - 3.4\% &= 1.2\% \\ Okun's\ Law &: -2(1.2) = -2.4\%\end{aligned}$$

24. If the labor force is 200 and the initial level of output is 500, what is the new level of output and the new level of employment?

    Answer: The change in output is

    $$\hat{X} = -2(0.046 - 0.034) = -0.024$$

    which implies that the new output is

    $$500 - (0.024)500 = 488.0$$

25. Using Okun's law and assuming the share of labor is constant, fill in the following table, where w is the nominal wage:

time	$\bar{L}$	L	u	X	X/L	$\hat{\rho}$	w	$\hat{p}$
1	100.0	92	.08	400	4.35	–	1	.03
2	100.5	93						.02
3	101.0	95						.01
4	101.5	97						.05
5	102.0	99						.04

Answer:

time	$\bar{L}$	L	u	X	X/L	$\hat{\rho}$	w	$\hat{p}$
1	100	92	0.08	400	4.35	-	1	.03
2	100.5	93	0.07	404.3	4.35	0	.02	.02
3	101.0	95	0.06	416.6	4.39	0.01	.02	.01
4	101.5	97	0.04	429.1	4.42	0.01	.06	.05
5	102.0	99	0.03	441.7	4.46	0.01	.05	.04