1 Tangency Condition

1.1 Three functions

These functions can serve as both utility or production functions

$$U = Ax_1^{\beta} x_2^{1-\beta}$$

Take the total differential

$$dU = A\beta x_1^{\beta-1} x_2^{1-\beta} dx_1 + A\beta x_1^{\beta} x_2^{1-\beta-1} dx_2$$

and set this equal to zero

$$0 = A\beta x_1^{\beta-1} x_2^{1-\beta} dx_1 + A(1-\beta) x_1^{\beta} x_2^{1-\beta-1} dx_2$$

divide by $Ax_1^{\beta} x_2^{1-\beta}$

$$0 = \frac{A\beta x_1^{\beta-1} x_2^{1-\beta}}{Ax_1^{\beta} x_2^{1-\beta}} dx_1 = \frac{-A(1-\beta) x_1^{\beta} x_2^{-\beta}}{Ax_1^{\beta} x_2^{1-\beta}} dx_2$$

and simplify

$$0 = \frac{\beta}{x_1} dx_1 = -\frac{(1-\beta)}{x_2} dx_2$$

and solve for the slope dx_2/dx_1

$$-\frac{\beta}{(1-\beta)} \frac{x_2}{x_1} = \frac{dx_2}{dx_1}$$

This is the slope of the indifference curve. The tangency condition is to set this equal to the budget constraint

$$B = px_1 + x_2$$

slope

For the *Cobb-Douglas,* the tangency condition is

$$p = \frac{\beta}{(1-\beta)} \frac{x_2}{x_1}$$

For the producer, the isoquant is

$$\begin{aligned} Q &= K^{\beta} L^{1-\beta} \\ dQ &= \beta K^{\beta-1} L^{1-\beta} dK + (1-\beta) K^{\beta} L^{1-\beta-1} dL \\ 0 &= \beta K^{\beta-1} L^{1-\beta} dK + (1-\beta) K^{\beta} L^{-\beta} dL \end{aligned}$$

And now substitute the production function back in

$$0 = \beta \frac{Q}{K} dK + (1-\beta) \frac{Q}{L} dL$$

The slope of the isoquant is then

$$\frac{dK}{dL} = -\frac{(1-\beta)}{\beta}\frac{K}{L} = -\frac{w}{p}$$

where $-w/p$ is the slope of isocost. This could also be written

$$0 = MPP_K dK + MPP_L dL$$

and

$$\frac{dK}{dL} = \frac{MPP_L}{MPP_K} = \frac{w}{p}$$

For the *square root function*, we have $\beta = 1/2$

$$p = \frac{x_2}{x_1}$$

1.2 Additive Natural Log

$$U = A_1 \ln(x_1) + A_2 \ln(x_2)$$

Take the total derivative

$$dU = 0 = A_1 \frac{dx_1}{x_1} + A_2 \frac{dx_2}{x_2}$$

$$A_1 \frac{dx_1}{x_1} = -A_2 \frac{dx_2}{x_2}$$

For the *Additive natural log function*, the tangency condition is

$$p = \frac{A_1}{A_2}\frac{x_2}{x_1}$$

2 General expression

$$U = f(x, y)$$

Again take the total differential

$$dU = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

and set this equal to zero

$$0 = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

and solve

$$\frac{\partial f}{\partial x}dx = -\frac{\partial f}{\partial y}dy$$

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}/\frac{\partial f}{\partial y}$$

So for a production function

$$\frac{w}{r} = \frac{MPP_L}{MPP_K}$$

where $MPP_L = \partial Q / \partial L$ and $MPP_K = \partial Q / \partial K$. If this a utility function then

$$p = \frac{MU_1}{MU_2}$$

3 More advanced material

This can be applied to more complex functions such as the CES

$$Q = A \left[\delta K^{-\rho} + (1-\delta) L^{-\rho} \right]^{-1/\rho}$$

(1) linearly homogeneous
(2) ρ is the substitution parameter
(3) δ is the distribution parameter

We saw above that to get the tangency condition, we could just take the ratio of marginal products of the factors of production. The first step is to differentiate using the chain rule for exponents and then substitute the original equation back in:

$$\frac{\partial Q}{\partial L} = (-1/\rho)\, A \left[\delta K^{-\rho} + (1-\delta) L^{-\rho} \right]^{(-1/\rho - 1)} \left[-\rho (1-\delta) L^{-\rho-1} \right]$$

This looks bad until we see that lurking inside the derivative is something that is similar to Q. To see that we note that the exponent -1 just means divide:

$$\frac{\partial Q}{\partial L} = -\frac{A \left[\delta K^{-\rho} + (1-\delta) L^{-\rho} \right]^{-1/\rho}}{\left[\delta K^{-\rho} + (1-\delta) L^{-\rho} \right]} \frac{\left[-\rho (1-\delta) \right]}{\rho L^{\rho+1}}$$

Which can then be expressed as

$$\frac{\partial Q}{\partial L} = -\frac{Q}{\left[\delta K^{-\rho} + (1-\delta) L^{-\rho} \right]} \frac{\left[-\rho (1-\delta) \right]}{\rho L^{\rho+1}}$$

Solving the original equation

$$\left[\delta K^{-\rho} + (1-\delta) L^{-\rho} \right] = \left(\frac{A}{Q} \right)^{\rho}$$

and substituting this in, we have

$$\frac{\partial Q}{\partial L} = -\frac{(Q/L)^{\rho+1}}{A^{\rho}} \frac{\left[-\rho (1-\delta) \right]}{\rho}$$

$$\frac{\partial Q}{\partial L} = A^{-\rho}(1-\delta)\,(Q/L)^{\rho+1}$$

Now do the same for capital

$$\frac{\partial Q}{\partial K} = A^{-\rho}\delta\,(Q/K)^{\rho+1}$$

So that the tangency condition is

$$\frac{dK}{dL} = -\frac{A^{-\rho}(1-\delta)\,(Q/L)^{\rho+1}}{A^{-\rho}\delta\,(Q/K)^{\rho+1}} = -\frac{w}{r}$$

or

$$\frac{(1-\delta)\,(K/L)^{\rho+1}}{\delta} = \frac{w}{r}$$