1 Vector Spaces and Matrix Notation

Definition 1 *A* matrix*: is rectangular array of numbers with n rows and m columns.*

Example 1 *a.* $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ *b.* $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$ *c.* $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

- The first is square with $n = 2$ and $m = 2$, the second is rectangular, $n = 2$ and $m = 3$. The third is written with its row-column index $a_{row,col}$ explicit so that a_{ij} is the *element* in the *ith* row and *jth* column.

- The only restriction on n and m is that the be integers, i.e., whole numbers greater than or equal to one. There are some important special cases

Definition 2 *A* scalar *is matrix with n and m both equal to one. If $n = 1$ but m is greater than one 1, the matrix is called a* row vector *and is written $X = \{x_j\}$ with j as the column index.. If $m = 1$ but n is greater than one 1, the matrix is called a* column vector *and is written $X = \{x_i\}$ with i as the row index.*

Definition 3 *A* transpose *of a matrix reverses its row column index*

$$M^T = \{m_{ji}\}$$

the transpose of a row vector is a column vector and vice-versa.

Definition 4 *A* linear combination *Z of vectors X and Y is*

$$Z = aX + bY$$

where a and b are given weights.

$$\begin{aligned} [M] \begin{matrix} a \\ b \end{matrix} &= Z \\ m_{11}a + m_{12}b &= Z_1 \\ m_{21}a + m_{22}b &= Z_2 \end{aligned}$$

Definition 5 *2 vectors are said to be* independent *if $Z = 0$ implies that both a and b are both zero. In other words, it is impossible to locate the origin with non-zero weights.*

Definition 6 *A* 2-dimensional vector space *is* spanned *by a linear combination of 2 independent vectors.*

- The Keynesian model can be expressed in matrix notation

$$\begin{aligned} C + I &= Y \\ \bar{C} + cY &= C \end{aligned}$$

$$\begin{aligned} C - Y &= -I \\ -C + cY &= -\bar{C} \end{aligned}$$

Definition 7 *or:*

$$\begin{array}{c} 1 \\ -1 \end{array} C + \begin{array}{c} -1 \\ c \end{array} Y = \begin{array}{c} -I \\ -\bar{C} \end{array}$$

1.1 Operations

Definition 8 *Scalar multiplication. The product of a scalar z and a matrix $A = \{a_{ij}\}$ is $zA = B\{za_{ij}\}$*

Note the convenient way we have represented the matrix A as the row-column index on the same, but lowercase letter, a_{ij}. The matrix B is the product and is the same size a A was before it was multiplied.

Example 2 *Let $z = 3$; and $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. We then have $zA = B$ where $B =$ $\begin{bmatrix} 6 & 3 \\ 0 & 9 \end{bmatrix}$*

Definition 9 *Matrix multiplication. The product of two matrices, A and B is equal to C under some restrictive conditions. First A and B must "*conform*" to multiplication. This means that column index of A must be equal to the row index of B. The dimension of C is then the row index of A and the column index of B. Second, the multiplication must follow a specific pattern. To get the first element of C, that is c_{11} take the first row of A and multiply it element-by-element times the first column of B and the add up the results. To get the get the second element of C, that is, c_{12}, take the first row of A and multiply it by the second column of B, element by element, add up the results. The element c_{ij} is then*

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

Example 3 *The product of the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\begin{bmatrix} 5 & 8 \\ 9 & 12 \end{bmatrix}$.*

The matrices conform since both are 2×2 and thus the column index of the first matrix is equal to the row index of the second. The product matrix has the row index of the first and the column index of the second.

Example 4 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix}$

Example 5 $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -2 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -12 & 12 \\ 14 & -6 \end{bmatrix}$

1.2 Properties of Matrix Multiplication

1.2.1 Vector Spaces are different from R^n

1. Matrix multiplication is finicky about the *order* of multiplication. While we normally think that $ab = ba$, it is *not* true for matrices (except for some special cases). If we reverse the order of multiplication above, then we get $\begin{bmatrix} 2 & 7 \\ 6 & 15 \end{bmatrix}$, an entirely different matrix. We use the terms *postmultiply* to indicate multiplication from the right and *premultiply* to indicate multiplication from the left.

2. It is possible to multiply rectangular matrices as long as they conform and the result if another rectangular matrix. In the example above, matrices a and b do conform because a is a 2×2 and b is a 2×3. The product matrix c is a 2×3. Reversing the order of matrix multiplication in this case does not work because they do not conform; the column index of b is 3 and this does not agree with the row index of a.

3. Two row vectors cannot be multiplied times each other, nor can two column vectors. A special case of matrix multiplication is a row vector multiplied times a column vector. Since the row vector has a row index of one and a the column has column index of one the product is a scalar or 1×1 matrix. This multiplication is called a *scalar, inner or dot product.*

4. Suppose $A = (a_{ij})$ is $m \times n$ and $B = (b_{jt})$ is $n \times k$, $t = 1, 2, ..., k$. Then

   (a) $(AB)\,C = A\,(BC)$ Associative Law of Multiplication

   (b) $A\,(B + C) = AB + AC$ Left Distributive Law of Matrix Multiplication over Addition

   (c) $(B + C)\,A = BA + CA$ Right Distributive Law of Matrix Multiplication over Addition

   (d) For any scalar r, $r\,(AB) = (rA)\,B = A\,(rB)$ Associative Law of Scalar and Matrix Multiplication

1.3 Determinants

If a matrix is *square* (that is, if it has the same number of rows as columns), then you can assign to it a number called its *determinant.* Determinants can be used to solve matrix equations. They are also useful in determining whether a matrix has an inverse, without actually going through the process of trying to find its inverse.

1.4 Determinant of a 2 x 2 Matrix

Let M be the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the *determinant* of M, denoted $|M|$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is given by

$$|M| = ad - bc$$

1.5 The Determinant of a Square Matrix

Now let A be an $n \times n$ matrix.

1. The *minor* M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and the jth column of A.
2. The *cofactor* is a *signed minor.* For A_{ij} of the element a_{ij} is

$$A_{ij} = (-1)^{i+j} M_{ij}$$

If A is an $n \times n$ matrix, then the determinant of A is obtained by multiplying each element of the first row by its cofactor, and then adding the results. In symbols,

$$\begin{aligned} |A| &= \begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{vmatrix} \\ &= a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} \end{aligned}$$

1.6 Identity Matrix

The $n \times n$ *identity* matrix I has ones down the main diagonal (upper-left corner to lower-right corner) and zeroes elsewhere. The 3×3 identity matrix, for example, is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The *inverse* of an $n \times n$ matrix A is an $n \times n$ matrix B satisfying $AB = I$.

1.7 Matrix inversion

1. The *inverse* of an $n \times n$ matrix A is an $n \times n$ matrix B satisfying $AB = I$.

2. We can use matrix algebra to solve the *linear system*

$$AX = b$$

by premultiplying by A^{-1}

$$A^{-1}AX = A^{-1}b$$

where A^{-1} is the *inverse matrix* for matrix A.

3. For A to have an inverse, it must

   (a) have the same number of rows and columns

   (b) have rows and columns that all linearly independent (sometimes known as full rank). This can be determined by the *Invertibility Criterion* which says that matrix A has an inverse if and only if $|A| \neq 0$.

   (c) Square matrices that do not have inverses are called **singular**

4. To invert a $2x2$ matrix, $\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$ we follow a set of rules:

   (a) Calculate the *determinant*

   (b) Calculate the *adjoint matrix.* For each entry a_{ij} of A, let A_{ij} be the cofactor of a_{ij}. The adjoint matrix of A is defined to be the $n \times n$ matrix whose entry in the i^{th} row and j^{th} column is A_{ji}. (note transpose)

   (c) Divide the adjoint by the determinant. If A is a nonsingular square matrix, then

   $$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A$$

   (d) Check to see that the product of the matrix and its inverse gives the identity matrix.

Example 6

$$\begin{aligned} A &= \begin{pmatrix} 5 & 6 \\ 8 & 7 \end{pmatrix} \\ \det A &= -13.0 \\ adj\,A &= \begin{pmatrix} 7 & -6 \\ -8 & 5 \end{pmatrix} \\ A^{-1} &= = \begin{pmatrix} -\frac{7}{13} & \frac{6}{13} \\ \frac{8}{13} & -\frac{5}{13} \end{pmatrix} \end{aligned}$$

1.8 Cramer's Rule

If a system of n linear equations in the n variables $a_1, x_2, ..., x_n$ is equivalent to the matrix equation $DX = B$, and if $|D| \neq 0$, then its solutions are

$$x_1 = \frac{|D_{x_1}|}{|D|}, \quad x_2 = \frac{|D_{x_2}|}{|D|}, ..., \quad x_n = \frac{|D_{x_n}|}{|D|}$$

where D_{x_i} is the matrix obtained by replacing the ith column of D by the $n \times 1$ matrix B.

1.9 Wassily Leontief, 1906-1999

- Wassily Leontief father of input-output analysis.
- Input-output was partly inspired by the Marxian and Walrasian analysis of general equilibrium via interindustry flows - which in turn were inspired by Quesnay's Tableau Economique
- Input-output analysis has been a mainstay of economics and economic policy and planning throughout the world for the past half-century.
- Raised in Russia, Leontief obtained his Ph.D. in Berlin.
- Formal input-output analysis developed at Harvard after 1932 to begin constructing an empirical example of his input-output system - an effort which gave rise to his 1941 classic, *Structure of American Industry.*
- Leontief followed up this work with a series of classical papers on input-output economics (collected in 1966).
- Input-output was novel and inspired large-scale empirical work. It has been used for economic planning throughout the world, whether in Western, Socialist or Third World countries.
- The structure of input-output (albeit with some critical differences) was employed by Piero Sraffa and the Neo-Ricardians in the 1960s to resurrect the theories of Ricardo and Marx.
- Leontief's contributions to economics were not limited to input-output.

- His 1936 article on "composite commodities" made him, together with Hicks, the father of that famous microeconomic theorem. His early reviews of Keynes's General Theory (1936, 1937, 1947, 1948) were important stepping stones to the Neo-Keynesian synthesis's stress on fixed nominal wages in interpreting Keynes's theory.

- 1933 article on the analysis of international trade is still relevant today and his 1946 contribution on the wage contract outlined what is now a classical application of the principal-agent model before that term was invented.

- 1953 found that US exported labor-intensive rather than capital- intensive goods - the "Leontief Paradox" - which brought into question the validity of the conventional factor-proportions theory of international trade.

- Taught at Harvard through the 1930s and developed much of post-war economics

- Leontief moved to the C.V. Starr Center at New York University.

- Critical of the misuse of mathematics and quantitative methods and the lack of relevance and realism in its theorizing

- Leontief won the Nobel memorial prize in 1973.

1.10 The Leontief Inverse

1. Input-output matrix A has a Leontief inverse

$$(I-A)^{-1}$$

   that results from the solution of the material balance

$$\begin{aligned} X &= AX + F \\ (I-A)X &= F \\ X &= (I-A)^{-1}F \end{aligned}$$

2. The inverse is given by

$$X = (I-A)^{-1}F = F + AF + A^2F + ... + A^nF$$

   where n needs not be much larger than 5 for a "good" approximation.

3. To see this, note that

$$(I-A)(I-A)^{-1} = I$$

should be just the identity matrix, so that

$$\begin{aligned}(1-A)(I-A)^{-1} &= (I-A)(I+A+A^2+...+A^n) \\ &= (I+A+A^2+...+A^n)-(A+A^2+...+A^{n+1}) \\ &= I-A^{n+1}\end{aligned}$$

so that if $\lim_{n\to\infty} A^{n+1}=0$, the approximation is valid.

4. For $\lim_{n\to\infty} A^{n+1}=0$ the matrix A must be *productive.* A productive matrix is one for which the maximal eigenvalue is less than one. More on this in an advanced course.

1.11 Duality

We can show the fundamental duality of input-output accounting. On the *output side* we have the material balance equation

$$X = AX + F$$

Note that in principle no prices are involved, only quantities. Summation across row of the input-output matrix can thought of as denominated in *physical units.* In practice the material balance equation still has to involve aggregation into units of similar quantities for each sector and thus prices are involved. This equation holds across the rows. Down the columns we have

$$PX = PAX + V$$

where PX is the gross value of production and V is a scalar (why). Now premultiply the material balance by P premultiply the material balance by P.

$$PX = PAX + PF$$

Subtract the last two equations

$$PF = V$$

That is the value of final demand is equal to the value added.

2 Exercises

Example 7 $A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.5 & 0.2 \end{bmatrix}$ *with* $F = \begin{bmatrix} 150 \\ 200 \\ 210 \end{bmatrix}$. *Determine the total demand for industries 1,2 and 3 given the matrix of technical coefficients and final demand vector F. Solution: First form* $(I-A)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.5 & 0.2 \end{bmatrix}$$

$= \begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.3 \\ -0.3 & -0.5 & 0.8 \end{bmatrix}$, *determinant: . Next form the adjoint matrix* $\begin{bmatrix} 0.57 & 0.34 & 0.27 \\ 0.41 & 0.58 & 0.32 \\ 0.47 & 0.49 & 0.6 \end{bmatrix}$ *and finally divide by the determinant* 0.239 *to get the Leontief inverse*

$$(I - A)^{-1} = \frac{1}{0.239} \begin{bmatrix} 0.57 & 0.34 & 0.27 \\ 0.41 & 0.58 & 0.32 \\ 0.47 & 0.49 & 0.6 \end{bmatrix}$$

$= \begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix}$. *The last step is to premultiply the F vector by the Leontief inverse* $\begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix} \begin{bmatrix} 150 \\ 200 \\ 210 \end{bmatrix} = \begin{bmatrix} 879.5 \\ 1023.9 \\ 1232.2 \end{bmatrix}$

Example 8 *A massive new public works program is initiated. Compute the change in total production if final demand increases by 40,20 and 25 respectively. Solution: We have* $\Delta X = (I-A)^{-1}\Delta F$. $\begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix}$ $\begin{bmatrix} 40 \\ 20 \\ 25 \end{bmatrix} = \begin{bmatrix} 152.09 \\ 150.63 \\ 182.43 \end{bmatrix}$

Example 9 *Let the direct labor employed per unit of output be given by* $L = \begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix}$. *Compute the employment impact of the new project. Solution: We have change in employment equal to the labor required by the change in final demand; that is:* $L\Delta X = L(I-A)^{-1}\Delta F$. $\begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix} \begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix}$ $\begin{bmatrix} 40 \\ 20 \\ 25 \end{bmatrix} = 107.74$. *This amounts to an increase of* $\begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix} \begin{bmatrix} 879.5 \\ 1023.9 \\ 1232.2 \end{bmatrix} =$ $107.74/699.15 =: 0.1541$ *or a 15% increase in employment. The denominator, 699.15 is given by* $L(1-A)^{-1}F = LX = \begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix} \begin{bmatrix} 879.5 \\ 1023.9 \\ 1232.2 \end{bmatrix} = 699.15$