An Essay on e

- Consider the function

$$f(m) = (1 + \frac{1}{m})^m$$

and evaluate that function at various levels of m.

$$\begin{aligned} f(1) &= (1+\frac{1}{1})^1 = 2 \\ f(2) &= (1+\frac{1}{2})^2 = 2.25 \\ f(3) &= (1+\frac{1}{3})^3 = 2.3707 \\ &\vdots \\ \lim_{m\to\infty}(1+\frac{1}{m})^m &= 2.71828 = e \end{aligned}$$

- This relationship has an immediate interpretation: let the interest rate be 100% and principal be equal to one dollar: The value of the dollar at the end of one year is then:

$$f(1) = (1 + \frac{1}{1}) = 2$$

We say that this interest is compounded *annually*. If we wish to compound *semi*-annually we would have *two periods and half the interest rate:*

$$f(2) = (1 + 0.5) = (1 + \frac{1}{2})^2 = \$2.25$$

- For three periods and a third of the interest rate:

$$f(3) = (1 + 0.333) = (1 + \frac{1}{3})^3 = \$2.3707$$

- Now take the limit of this process:

$$\lim_{m\to\infty}(1 + \frac{1}{m})^m = 2.71828 = \$e$$

This is called *continuous compouding* and gives an effective interest rate of e.

- Consider what this means in terms of biological growth rates. Say a species of bacteria can clone itself but it takes time to do so, a gestational period. Define that period as one unit of time. Note that the growth process can be described as:

$$\Delta X \backsim X$$

the *change* in the population X is proportional to the *level* of the population. It follows that if there were more bacteria around, the change ΔX would be proportionally greater. Now let the bacteria be *divisible.* That is a "half bacterium" can also clone itself in one period and *in half a period it clones half itself* or a quarter of the bacterium. Let this be the smallest bacterium than can clone itself. The gestational period has not changed but it has been subdivided to allow partial bacterium to reproduce themselves (Think of teenage mothers!) Since the change ΔX is still proportional to the level X and the level has increased by the divisibility then the change will be proportionally larger. The number of bacterium at the end of one period would be 2.25. Now make the smallest bacterium that can clone itself be 1/9 (or a third of a third of a bacterium). At the end of one period there will be even more (2.3707) but there are *diminishing returns* to this trick of lowering the cloning age and dividing the bacterium. Even if newly cloned fraction of a bacterium could immediately reproduce itself, the upper limit would be e, or 2.71828. Note that as we approach this limit, the gestational period *has not changed.* It still takes a full period for one bacterium to fully reproduce itself.

- Now to generalize to different interest or growth rate, write:

$$f(m)^{rt} = [(1+\frac{1}{m})m]^{rt}$$

and if we took the limit

$$\lim f(w)^{rt} = [\lim_{w\to\infty}(1+\frac{1}{w})]^{rt} = e^{rt}$$

Now let

$$w = \frac{m}{r}$$

for a given r.and note that

$$\begin{aligned} \lim f(m)^{rt} &= [\lim_{m\to\infty}(1+\frac{r}{m})^{m/r}]^{rt} \\ \lim f(m)^{rt} &= \lim_{m\to\infty}(1+\frac{r}{m})^{mt} = e^{rt} \end{aligned}$$

- Thus, a population of X_0 at time t would grow at rate r to

$$X_t = X_0 e^{rt}$$

at the end of t periods of time.

- In summary, e is the limit of the ratio of the final (end of the period) to the beginning level of X as the length of time required before X can begin to reproduce itself goes to zero.