Check your understanding: Hat Calculus[1]

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[1] Thanks to Diane Flaherty

The rules of hats are simple and similar to **logarithms**.

Rule 1 Multiplication

$$x = yz$$

where x, y and z are the *levels* of any three variables. The first rule says

$$\hat{x} = \hat{y} + \hat{z}$$

So if the levels are multiplied, then the hats are added.

Rule 2 Division

$$x = y/z$$

The second rule says

$$\hat{x} = \hat{y} - \hat{z}$$

So if the levels are divided, then the hats are subtracted.

Rule 3 Multiplication by a constant. Let

$$x = ay$$

be the *levels* of the two variable x and y with a constant. The third rule says

$$\hat{x} = \hat{y}$$

So if the level is multiplied by a constant, it drops out. *Exercise: prove that this rule is true using Rule 1.* Remark: since a could equal $1/b$ the same is true for division by a constant.

Rule 4 Exponents

$$x = y^a$$

be the *levels* of the two variable x and y with a constant. The fourth rule says

$$\hat{x} = a\hat{y}$$

So if the levels are raised to a constant exponent then, in this case, the constant does not disappear.

Rule 5. *Exponential growth*. Now there is a special case of the exponent rule when $y = e$, the base of the natural logarithm system:

$$Y = e^{gt}$$

In this case, $\hat{Y} = g$.

Rule 6. The rule of 69, 70 or 72. This is about doubling times. If the rate of growth is g (on average) to find the doubling time, divide 69 or 70 or 72 by g

$$t = 70/g$$

Which number do we use? If the process has a quick turn around, think bacteria multiplication in a petri dish, or a nuclear chair reaction, use 69. If slower, say human gestation of 9 monthsuse, 70 and even slower, economic growth determined by the accumulation of physical capital, use 72. (The answer is not wildly different no matter which one you choose.)

Note that like all applications of calculus, these rules only apply for "small" changes; for *large changes they are only approximations*. If the rate of growth is over, say, 10-15 percent, don't use these rules; you will be off.

There is no really addition rule. If $x + y = z$ then it is only approximately true that $\hat{z} = x/(x+y)\hat{x} + y/(x+y)\hat{y}$. The problem here is that the weights in the weighted average change. There is a special case, however, and that is when the weights don't change. This only happens when the rate of growth of x and y are the same and so, necessarily, the rate of growth of z must be the same too!

Worked examples

1. Nominal GDP grows at 6% but inflation is 5%. What is the approximate growth rate of real GDP?[2]

2. Output per worker is defined as

$$\rho = X/L$$

   where X is the real GDP and L is number of employed workers. Productivity usually grows at around 1%. If employment is up by 3% this year, what is the growth rate of real GDP?[3]

3. The rate of growth of the capital stock is 4 percent. How long will it take for the capital stock to double? [4]

4. The rate of inflation is $\hat{p}$ and it is equal to -2% per year. What rate of nominal growth in GDP would be require for real growth to be 3%?[5]

5. With the same rate of inflation as in the previous problem what rate of growth of *nominal* wages is required to keep real wages constant? [6]

6. Let the interest rate on a CD by 6 percent. How many month will it take for a deposit of 100 dollars to double?[7]

[2] 1%.

[3] We have

$$\begin{aligned} \hat{\rho} &= 1\% \\ \hat{L} &= 3\% \\ \hat{X} &= 4\% \end{aligned}$$

[4] Here use the 72/4 = 18 years.

[5] We have

$$\begin{aligned} Y &= pX \\ \hat{Y} &= \hat{p} + \hat{X} \\ 3\% &= 5 - 2 \end{aligned}$$

[6] We have

$$\begin{aligned} (w/p)_{hat} &= \hat{w} - \hat{p} \\ 0 &= \hat{w} + 2\% \\ \hat{w} &= -2\% \end{aligned}$$

[7] Here one must be careful: the *monthly rate of growth* is only 0.5 perent. Divided this into 70 gives 140 months, not years. This corresponds to almost 11 years and 8 months.

7. If the share of labor, also known a s$(1-\beta)$ is defined as

$$\sigma = \frac{wL}{pX}$$

what is the meaning of L, w, w/p and X/L? If $\hat{\sigma} = 0$, what is the relationship between the real wage and productivity growth?[8]

8. If the rate of growth of the share of labor is zero and the rate of employment growth is 2%, what is the rate of growth of the real wage required for 4% growth rate of real GDP?

$$\begin{aligned} 0 &= \hat{w} + \hat{L} - \hat{p} - \hat{X} \\ &= \hat{w} - \hat{p} + 2 - 4 \\ \hat{w}_r &= \hat{w} - \hat{p} = 2\% \end{aligned}$$

Hence the real wage w/p grows by 2%.

9. If the rate of inflation is 8% in the previous problem what must be the rate of growth of nominal wages?

$$\begin{aligned} \hat{w}_r &= \hat{w} - 8 = 2\% \\ \hat{w} &= 10\% \end{aligned}$$

10. If the labor force is 200 and the initial level of output is 500, what is the new level of output and the new level of employment?[9]

11. Let the growth in employment be zero. If the economy achieves a real growth rate of 3% with 5% inflation and nominal wages increase by 4% what happens to the share of labor?[10]

12. Let capacity output be Q and define *capacity utilization* as

$$\mu = X/Q$$

Say μ is 0.85. If Q grows by 3.5% per year, but output grows by

3% what happens to capacity utilization? [11]

13. Fill in the following table: X is real GDP and Y is nominal GDP.

time	X	Y	$\hat{p}$
1	100	100	0.03
2	101	103	0.02
3	104	110	0.04
4	97	108	0.05
5	112	115	−0.09

[8] Let L = employment; w = wage rate w/p = real wage and X/L = productivity. The real wage must grow at the same rate as productivity.

[9] The change in output is

$$\hat{X} = \Delta X/X = (-2)(0.046 - 0.034) = -0.024$$

which implies that the new output is

$$500 - 0.024(500) = 488.0$$

[10]

$$\begin{aligned} \sigma &= \hat{w} + \hat{L} - \hat{p} - \hat{X} \\ &= 4\% + 0\% - 5\% - 3\% \\ &= -4\% \end{aligned}$$

Again, remember that the share of labor is itself a percentage.

[11] It falls by 0.5%.

14. What is the *average* rate of growth of real GDP.[12]

[12] The average rate of growth of GDP is the g that gives:

$$100(1+g)^4 = 112$$

Solving this expression for g we have:

$$g = (\frac{112}{100})^{\frac{1}{4}} - 1 = 0.0287$$

15. Why can we not take the yearly rates of growth of real GDP and average them?[13]

[13] Because of compounding each year's average growth rate applies to the previous year's level which includes the growth from all years before. The average is .02882. This is very close, but larger than the true average.

16. Fill in the following table assuming a rate of growth of the labor force of 1% and a *constant* rate of unemployment.

time	X	L	$\hat{\rho}$
1	100	100.0	0.03
2	101	101.0	0.00
3	104	102.0	0.020
4	107	103.0	0.019
5	112	104.1	0.036