Check your understanding: Exponent rules[1]

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[1] Thanks to Josh Audette

Exponents

In the expression. a^n, a is the *base* and n is the *exponent*, also known as a *power*.

1. Compute the value of 3^2, using Excel.[2]
2. Compute the value of 3^3, using Excel.[3]
3. Compute the value of 3^n, using Excel.[4]

[2] $3^2 = 3 \times 3 = 9$.

[3] $3^3 = 3 \times 3 \times 3 = 27$.

[4] $3^3 = 3 \times 3 ... \times 3$ n times.

A negative exponent means one has to take 1 over the base and raise result to the exponent.

1. Compute the value of 3^{-2}, using Excel.[5]
2. Compute the value of 3^3, using Excel.[6]
3. Compute the value of 3^n, using Excel.[7]
4. If n is large, even 4 or 5, what is the approximate value of a^{-n}, using Excel.[8]

[5] $(3)^{-2} = 1/3 \times 1/3 = 1/9$.

[6] $3^{-3} = 1/3 \times 1/3 \times 1/3 = 1/27$.

[7] $3^{-3} = 1/3 \times 1/3 ... \times 1/3$ n times.

[8] If $a > 1$, $a^{-n} \to 0$ when a is large. Try this. The larger the a the fast the term goes to zero when n grows large. When a is less than zero, the $a > 1$, a^{-n}, explodes *exponentially*.

Exponents of fractions are tricky to compute. Note that Excel gives $2/3^4 = 0.0247$ but this would be wrong if the problem asked for 2/3 raised the fourth power. This is $(2/3)^4 = 0.1975$, a very different number. Here the parentheses are absolutely necessary to get the right answer.

It seems that when working with exponents, multiplication is associated with addition and division with subtraction. Operations seem to be bumped down a notch. This turns out to be true.

Product rules

We have to look for patterns here to know what to do. If the *base* is the same we have $a^n a^m = a^{(n+m)}$, using Excel. If the *exponent* is the same, we have $a^n b^n = (ab)^n$.

1. Compute the value of $3^2 3^3$, using Excel.[9]
2. Compute the value of $3^2 4^3$, using Excel.[10]
3. Compute the value of $3^2 4^2$, using Excel.[11]

[9] $3^2 3^3 = 3^{(2+3)} = 3^5 = 243$.

[10] We cannot use the product rule here since the bases, 3, are no longer the same. $3^2 4^3 = 576$.. Each term must be evaluated separately.

[11] $3^2 4^2 = (34)2 = 144$.

Quotient rules

Again, we have to look for patterns here to know what to do. If the *base* is the same we have $a^n/a^m = a^{(n-m)}$; notice that the plus has changed to a minus in the exponent. If the *exponent* is the same, we have $a^n/b^n = (a/b)^n$.

1. Compute the value of $2^5/2^3$, using Excel.[12]
2. Compute the value of $4^3/2^3$, using Excel.[13]

[12] Here the bases are the same so we can just subtract the exponents: $2^5/2^3 = 2^{(5-3)} = 2^2 = 4$.

[13] Here the exponents are the same so we can just divide the bases. $4^3/2^3 = (4/2)^3 = 8$.

Power rule

If a base raised to an exponent and the result is *again* raised to second exponent, we have $(a^n)^m = a^(nm)$. Powers are multiplied, consistent with the idea that in exponents algebraic operations are shifted down a notch.

1. Compute the value of $(2^3)^2$, using Excel.[14]
2. Compute the value of $(2^3)^{1/2}$, using Excel.[15]

[14] $(2^3)^2 = 2^(32) = 64$.

[15] $(2^3)^(1/2) = 2^{(3/2)} = 2.828$. Careful with parens!

Radicals

Radicals are really no different from the power rule. The are simply fractions and behave in the same way. For example $\sqrt{a} = a^{1/2}$ and so $\sqrt[n]{a} = a^{1/n}$. Once we convert the radical sign to an exponent the power rule can be applied.

1. Compute the value of $(\sqrt{2})^3$, using Excel.[16]
2. Compute the value of $(\sqrt[3]{2})^4$, using Excel.[17]

[16] $\sqrt{2}^3 = 2^{(3/2)} = 2.828$

[17] $(\sqrt[3]{2})^4 = (2^{(1/3)})^4 = 2^{(4/3)} = 2.52$ Again, careful with parens!

Cobb-Douglas

The Cobb-Douglas equation is a common example of an exponential function, used as a production function as in $Q = K^\beta L^\beta$, where Q output, K is capital and L is labor, or in a utility function $U = x^\alpha y^\alpha$ where U is utility, x is first good and y is the second good. The reason the Cobb-Douglas equation is so popular is that the marginal products and marginal utilities are so easy to calculate and remember.

The marginal products of capital, m_K and labor m_L

$$m_K = \beta Q/K$$
$$m_L = (1-\beta)Q/L$$

and the marginal utilities of x and y are

$$m_x = \alpha U/x$$
$$m_y = (1-\alpha)U/y$$

1. Let $\beta = 1/3$. Compute the value of the value of the marginal productivity of labor if $K = 20$ and $L = 10$.[18]

2. Let $\beta = 1/4$. Compute the value of the value of the marginal productivity of capital if $K = 2$ and $L = 4$.[19]

3. Let $\beta = 0.28$. Compute the output per worker if $K = 120$ and $L = 64$.[20]

4. A consumer has $x = 10$ and $y = 20$ and a $\alpha = 0.3$. Compute the marginal utility of an increase in x.[21]

5. Assume that the level of $K = 25$. If output is 40, and $\beta = 0.35$, compute the quantity of labor hired.[22]

[18] $m_L = (1-\beta)Q/L = m_L = (1-\beta)K^{\beta}L^{(1-\beta-1)}$ using the quotient rule. This can be further simplified to $m_L = (1-\beta)(K/L)^{\beta} = (2/3)(2)^{(1/3)} = 0.84$.

[19] $m_K = \beta Q/K = m_K = \beta K^{(\beta-1)}L^{(1-\beta)}$ using the quotient rule. This can be further simplified to $m_K = \beta(K/L)^{(\beta-1)} = (1/4)(1/2)^{(}-3/4) = 0.42$.

[20] Output per worker is $Q/L = K^{\beta}L^{(1-\beta)}/L = K^{\beta}L^{(1-\beta-1)} = (K/L)^{\beta} = (120/64)^{0.28} = 1.19$.

[21] $U = 10^{0.3}20^{0.7} = 16.245$.

[22] From $Q = K^{\beta}L^{(1-\beta)}$, solve for L. First divide by $K^{(1-\beta)}$ to find $Q/K^{\beta} = L^{(1-\beta)}$, using Excel. Next take the $1/(1-\beta)$ root of both sides to find: $(Q/K^{\beta})^{[1/(1-\beta)]} = L$ or $[(40/25)^{0.35}]^{[1/(1-0.35)]} = [40/(25^{0.35})]^{(1/0.65)} = 51.52$.