Check your understanding: Cobb Douglas Utility and Production Functions[1]

[1] Thanks to Farzad Ashouri

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Preliminaries

Since the consumer utility problem requires solving two equations and two unknowns simultaneously, lets first get some practice. The solutions are in the margin notes.

Solving two equations and two unknowns

1. Solve for x and y.[2]

[2] $x = 2, y = 1$

$$\begin{aligned} 4x + y &= 9 \\ 3x &= 6 \end{aligned}$$

2. Solve for x and y.[3]

[3] $y = 2.0, x = 1.75$.

$$\begin{aligned} 4x + y &= 9 \\ 3y &= 6 \end{aligned}$$

3. Solve for x and y.[4]

[4] $x = 4.25, y = 1.5$.

$$\begin{aligned} 4x + 2y &= 20 \\ 2x + 3y &= 13 \end{aligned}$$

4. Solve for x and y.[5]

[5] $x = 3.5, y = 3.0$.

$$\begin{aligned} 4x + 2y &= 20 \\ 2x + 3y &= 16 \end{aligned}$$

5. Solve for x and y.[6]

[6] $x = 6.0, y = 2.0$.

$$\begin{aligned} x + 2y &= 10 \\ 2x - 3y &= 6 \end{aligned}$$

Consumer's problem

The consumer's problem always involves solving the tangency condition (slope of the indifference curve equal slope of the budget constraint, p) with the budget constraint itself. Note that the tangency condition is

$$p = \frac{\alpha}{1-\alpha}\frac{y}{x}$$

the budget constraint is

$$B = px + y$$

After getting the solution for the demand for both goods, one can calculate the level of utility to see how well off the consumer is. This is given by

$$U = x^{\alpha} y^{1-\alpha}$$

. The utility function is Cobb-Douglas (CD) in all the problems that follow.

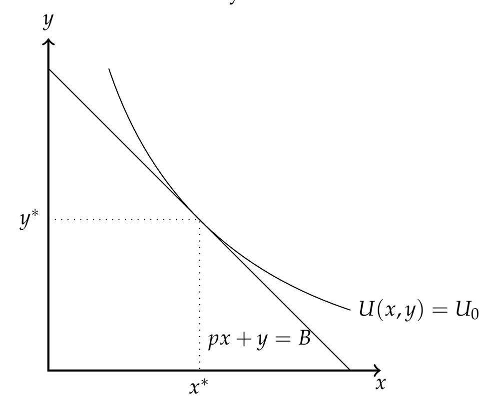


Figure 1: The consumer's problem. Equilibrium is determined by solving the tangency condition and the budget constraint simultaneously.

1. Let the price of good 1 be 4 and the price of good 2 be 1, as it always is in the problems. The solution for budget constraint with $B = 10$ is

$$4x + y = 10$$

Solve the following consumer's problem with $p = 3$ for the demand for x and the demand for y. The first is the budget constraint and the second is the tangency condition.

$$\begin{aligned} 3x + y &= 12 \\ 3 &= \frac{y}{x} \end{aligned}$$

[7]

Solve the following consumer's problem for x and y.[8]

$$\begin{aligned} \frac{3}{4}x + y &= 3 \\ \frac{3}{4} &= \frac{y}{x} \end{aligned}$$

2. Solve the following consumer's problem for x and y with $\alpha = 0.4$, $p = 1$ and $B = 2$.[9]

$$\begin{aligned} x + y &= 2 \\ 1 &= \frac{\alpha}{(1-\alpha)} \frac{y}{x} \end{aligned}$$

3. Let the budget be $B = 10$ and the relative price be $p = 0.5$. With a square root utility function, solve the demand for x and$y$.[10]

$$\begin{aligned} 0.5 &= \frac{y}{x} \\ 10 &= 0.5x + y \end{aligned}$$

4. From the previous problem solve for utility.[11]

[7] $y = 6.0, x = 2.0$

[8] $y = 1.5, x = 2.0.$

[9] $y = 1.2, x = 0.8.$

[10] $x = 10.0, y = 5.0.$

[11] $U = \sqrt{xy} = \sqrt{10(5)} = 7.0711.$

5. With a budget of $B = 10$ and the relative price be $p = 0.5$ solve the consumer's problem for x and y with $\alpha = 0.5$.[12]

$$\begin{aligned} 0.5 &= \frac{y}{x} \\ 10 &= 0.5x + y \end{aligned}$$

[12] $x = 10.0, y = 5.0$

6. From the previous problem solve for utility.[13]

[13] $U = 7.071$. Note that this is the same as the square-root utility function!

7. Solve for x and y with $\alpha = 0.3$, $B = 10$ and $p = 0.5$.[14]

$$\begin{aligned} 0.5 &= \frac{0.3}{0.7}\frac{y}{x} \\ 10 &= px + y \end{aligned}$$

[14] $x = 6.0, y = 7.0$.

8. From the previous problem solve for utility.[15]

[15] $U = 6.6837$.

9. Solve the same system for x and y with $\alpha = 0.7$ with $p = 0.5$.[16]

$$\begin{aligned} 0.5 &= \frac{0.7}{0.3}\frac{y}{x} \\ 10 &= px + y \end{aligned}$$

[16] $x = 14.0, y = 3.0$.

10. From the previous problem solve for utility.[17]

[17] $U = 8.8191$.

11. Solve the same system for x and y with $\alpha = 0.7$ and $p = 1$.[18]

$$\begin{aligned} 1 &= \frac{0.7}{0.3}\frac{y}{x} \\ 10 &= x + y \end{aligned}$$

[18] $x = 7.0, y = 3.0$

12. From the previous problem solve for utility.[19]

[19] $U = 5.4288$. The higher price makes the consumer worse off.

Producer's problem

The producer does not have a utility function, but rather seeks to maximize profits subject to a cost constraint. Mathematically the problem is *identical* to the consumer's problem. Only the interpretation of the equations and variables change.

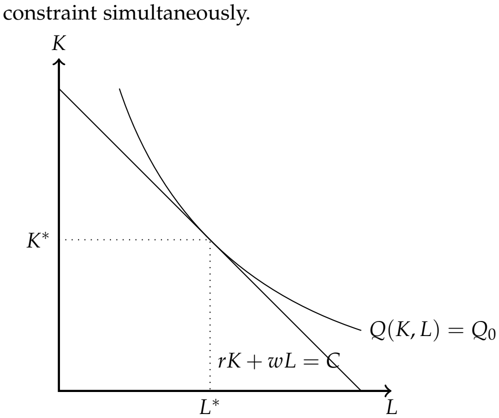
The production function is written $Q = K^{\beta}L^{1-\beta}$ where Q is output, K is the capital stock and L is the amount of labor employed. The parameter is β is a share parameter, the share of the return to capital in total product if both factors are *paid their marginal products*. The expression for the marginal product, m, is given (by calculus) as

$$m_K = \beta Q/K$$

and for labor

$$m_L = \beta Q/L$$

Figure 2: The producer's problem. Equilibrium is determined by solving the tangency condition and the cost constraint simultaneously.



Both show diminishing marginal productivity. The rational producer sets the *value of the marginal product* equal to the factor return, profit and the wage rate:

$$r = p\beta Q/K$$

where r is the rate of return on capital. For labor

$$w = p(1-\beta)Q/L$$

where w is the wage rate.

From these two equations, we have

$$rK/pQ = \beta$$

and for labor

$$wL/pQ = (1-\beta)$$

. These equations highlight the role of the parameter β, which determines the share of capital and labor in total output. In general these shares are approximately constant in most real economies and this makes the Cobb-Douglas production function a good approximation. There is also empirical support for constant returns to scale for many countries.

1. What this share of labor in the square-root production function?[20]
2. Is the square-root production function realistic?[21]
3. What is the output per worker if the production function is $Q = K^{\beta}L^{1-\beta}$ with $\beta = 0.25$ and the capital-labor ratio is 3?[22]
4. As development procedes, the capital-labor ratio increases. Assume that it rises by 40%. What is the change in output?[23]
5. Why is the percentage change in the previous problem so small?[24]

[20] $1-\beta = 0.5$

[21] No, usually the share of capital would be less than 50 percent in any actual economy. We use it to simplify the calculations.

[22] From $Q = K^{\beta}L^{1-\beta}$, we divide by L to get: $Q/L = K^{\beta}L^{1-\beta}/L$. Using the rules of exponents: $Q/L = K^{\beta}L^{1-\beta-1} = K^{\beta}L^{-\beta} = (K/L)^{\beta}$ which is usually written: $q = k^{\beta}$. Thus, $q = 3^{0.25} = 1.316$.

[23] Raising k by 40 percent is the same as multiplying by 1.4. So, we have $q = [3(1.4)]^{0.25} = 1.44$. The difference is then 0.123, which is only about 9.3%.

[24] This is because of *diminishing returns* to capital per unit of labor. Note that $\beta \times 40\%$ is 10% which is close to the 9.3 % calculated. For a smaller change in k, the formula $Deltaq/q = \hat{q} = \beta\Delta k/k = \hat{k}$ is approximately correct. We will cover the "hat" rules later in the course.