Check your understanding: General Equilibrium

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A solution is presented for a two-good, two-factor and one consumer general equilibrium model

Production

First define the production of both goods

$$Q_i = K_i^{\beta_i} L_i^{(1-\beta_i)} \tag{1}$$

where Q is output, K is the capital stock and L is the quantity of labor for $i = 1, 2$. The two factor demand equations

$$\bar{L} \geq L_1 + L_2 \tag{2}$$

where $\bar{L}$ is labor supply and L_i the demand for labor from each sector. The supply of capital $\bar{K}$,

$$\bar{K} \geq K_1 + K_2 \tag{3}$$

is greater than or equal to the demand for capital.

If the factor constraints bind, equations 2 and 3 hold with an equality; if not they are inequality constraints.

Consumption

There is a representative consumer with a Cobb-Douglas utility function, U, given by

$$U = Q_1^{\alpha} Q_2^{(1-\alpha)} \tag{4}$$

where α is a given parameter. The budget for this consumer it the entire GDP, Y given by

$$Y = pQ_1 + Q_2$$

Maximizing utility in equation 4 subject to the budget constraint gives the demand for each good.

$$Q_1 = \frac{\alpha}{p} \tag{5}$$

$$Q_2 = (1 - \alpha) \tag{6}$$

where p is the relative price.

Assumptions

1. Capital stocks, K_i are fixed and given.
2. square-root production functions
3. Labor constraints bind with given $\bar{L}$

Solution

Set the slope of the PPF to slope of the indifference curve of the consumer. This is the *tangency condition.* Solve this simultaneously with the equation for the PPF. The equation for the PPF is obtained by substituting equation 2 (as in equality) into production function for the second good

$$Q_2^2 = K_2(\bar{L} - L) \tag{7}$$

where for simplicity L_1 is written as L. From the production function for the first good

$$L = \frac{Q_1^2}{K_1} \tag{8}$$

Substituting L into the production function for the second good in equation 14 gives and expression for the PPF

$$Q_2^2 = K_2(\bar{L} - \frac{Q_1^2}{K_1}) \tag{9}$$

which gives the expression for the PPF in terms of parameters and parameters only. This expression can be differentiated to get the slope of the PPF. Taking differentials

$$2Q_2 dQ_2 = -\frac{2Q_1K_2}{K_1} dQ_1$$

or

$$\frac{dQ_2}{dQ_1} = -\frac{Q_1K_2}{Q_2K_1} \tag{10}$$

The slope of the indifference is given by differentiating 4. First take the differential

$$dU = \partial U/\partial Q_1 dQ_1 + \partial U/\partial Q_2 dQ_2$$

and since $dU = 0$ along the indifference curve

$$\frac{dQ_1}{dQ_2} = -\frac{\partial U}{\partial Q_2} / \frac{\partial U}{\partial Q_2}$$

where

$$\frac{\partial U}{\partial Q_1} = \frac{\alpha U}{Q_1}$$

$$\frac{\partial U}{\partial Q_2} = \frac{(1-\alpha)U}{Q_1}$$

Substituting into equation 10 to obtain the slope of the indifference curve

$$\frac{dQ_2}{dQ_1} = -\frac{\alpha}{(1-\alpha)}\frac{Q_2}{Q_1} \tag{11}$$

Setting the slope of the indifference in equation 11 to the slope of the PPF in equation 10

$$\frac{Q_1 K_2}{Q_2 K_1} = \frac{\alpha}{(1-\alpha)}\frac{Q_2}{Q_1} \tag{12}$$

Simplifying

$$\frac{Q_1^2}{K_1} = \frac{\alpha}{(1-\alpha)}\frac{Q_2^2}{K_2} \tag{13}$$

Now solve the tangency equation 13 with the equation for the PPF, 9

$$Q_2^2 = K_2(\bar{L} - \frac{\alpha}{(1-\alpha)}\frac{Q_2^2}{K_2})$$

Solve for Q_2

$$Q_2^2[1 + \frac{\alpha}{(1-\alpha)}] = K_2\bar{L}$$

getting a common denominator

$$Q_2^2[\frac{1}{(1-\alpha)} = K_2\bar{L}$$

The solution for Q_2 is then

$$Q_2 = \sqrt{(1-\alpha)\bar{L}K_2} \tag{14}$$

Solving equation 9 for Q_1^2

$$Q_1^2 = K_1(\bar{L} - Q_2^2/K_2)$$

and substituting 14

$$Q_1 = \sqrt{\alpha\bar{L}K_1} \tag{15}$$

Price and GDP

The price is given by equation by either the slope of the PPF or slope of the indifference curve, since in equilibrium they are the same. The slope of the indifference curve is the marginal rate of substitution

$$p = \frac{\alpha}{(1-\alpha)}\frac{Q_2}{Q_1} \tag{16}$$

but this can be simplified by substituting in equations 15 and 14

$$p = \frac{\alpha}{(1-\alpha)}\sqrt{\frac{(1-\alpha)\bar{L}K_2}{\alpha\bar{L}K_1}}$$

Simplifying

$$p = \sqrt{\frac{\alpha}{1-\alpha}\frac{K_2}{K_1}} \tag{17}$$

GDP is then just

$$Y = pQ_1 + Q_2 = \sqrt{\frac{\alpha^2 \bar{L} K_2}{(1-\alpha)}} + \sqrt{(1-\alpha)\bar{L}K_2} \tag{18}$$

Numerical example

Given the parameter values in table 1, the solutions are as follows:

$$Q_1 = \sqrt{0.3(148.225)10} = 21.087$$
$$Q_2 = \sqrt{(1-0.3)(148.225)5} = 22.777$$
$$p = \sqrt{\frac{0.3}{0.7}\frac{5}{10}} = 0.4629$$
$$Y = \sqrt{\frac{0.3^2(148.225)(5)}{0.7}} + \sqrt{(0.7)(148.225)5} = 32.538$$

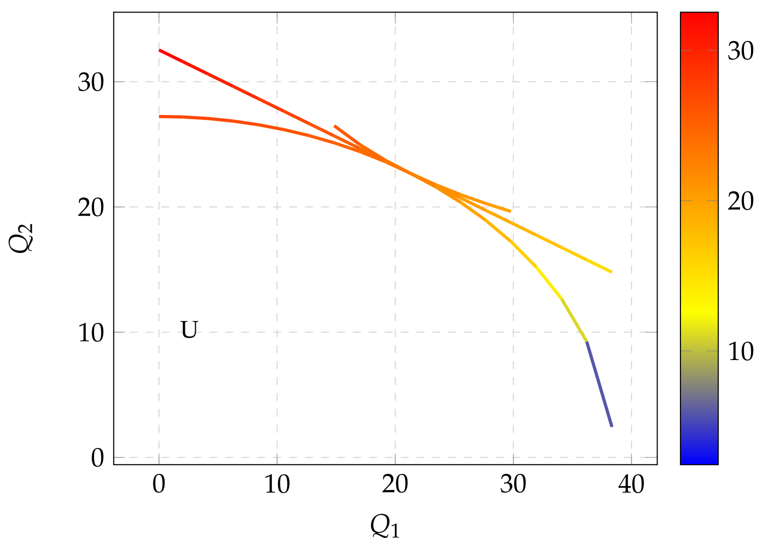

Figure 1: The PPF and Indifference Curves

Source: Author's calculations

Table 1: Parameter Values for the Numerical Example

Parameters		
α	Budget share of first good	0.3
$\bar{L}$	Labor supply	148.225
K_1	Capital stock for first sector	10
K_2	Capital stock for first sector	5