

Linear Programming Example

1. A country produces two basic goods steel and cotton. There are other goods, but the planners are not concerned with them. The price of cotton is 3 and of steel is 4. Steel requires 2.5 units of unskilled labor, 3 units of skilled labor and 1 unit of capital. Cotton requires 1 unit of unskilled labor, 3 units of skilled labor and 2 units of capital. The country has 20 units of skilled labor, 30 units of unskilled labor and 16 units of capital. Compute the optimal outputs of steel and cotton.

2. Solution: Set up the *primal* problem

$$\begin{aligned} \max pX \\ \text{s.t. } AX \leq b \\ X \geq 0 \end{aligned}$$

3. The constraints are

	<i>Cotton</i>	<i>Steel</i>
<i>Labor</i>	2.5	1
<i>Skilled Labor</i>	3	3
<i>Capital</i>	1	2

4. The data for the problem

$$A = \begin{bmatrix} 2.5 & 1 \\ 3 & 3 \\ 1 & 2 \end{bmatrix} \quad p = [3, 4] \quad b = \begin{bmatrix} 20 \\ 30 \\ 16 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

5. The non-negativity constraint is necessary to define the feasible region and precludes unreasonable solutions. The graph of the constraint in the less than or equal expression includes all the points on the line and interior to it. Here the X s are the *control* variables.
6. Next graph the constraints as if they were equalities. If any combination of production of cotton and steel is inside a constraint then we say there is *slack* in that constraint and so it does not *bind*.
7. Next plot the objective function

$$U = p_1x_1 + p_2x_2$$

for a range of values of U .

8. GDP is maximized at the intersection of two constraints, called the extreme point.
9. The solution (4,6) is obtained graphically and shown in figure 1. The optimal value of the objective function is 36.
10. Note that there are ten extreme point here (0,20),(0,10),(6,5),(10,0), (16,0),(0,8),(4,6),(6.67,3.34),(8,0) and (0,0). All are called *basic solutions* but only the last five are *basic feasible solutions* since they violate none of the constraints. Most of the time (not always) only one of the basic feasible solutions will be optimal. Note that at (6.67,3.34) for example, the value of the objective function is only 33.34, less $U = 36$.
11. Problems involving more than two variables cannot be solved graphically.
12. Computer programs cannot easily handle inequalities, so if we want to use Excel or some other program, the first step is to introduce *slack variables*. The A matrix becomes

$$A = \begin{bmatrix} 2.5 & 1 & 1 & 0 & 0 \\ 3 & 3 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

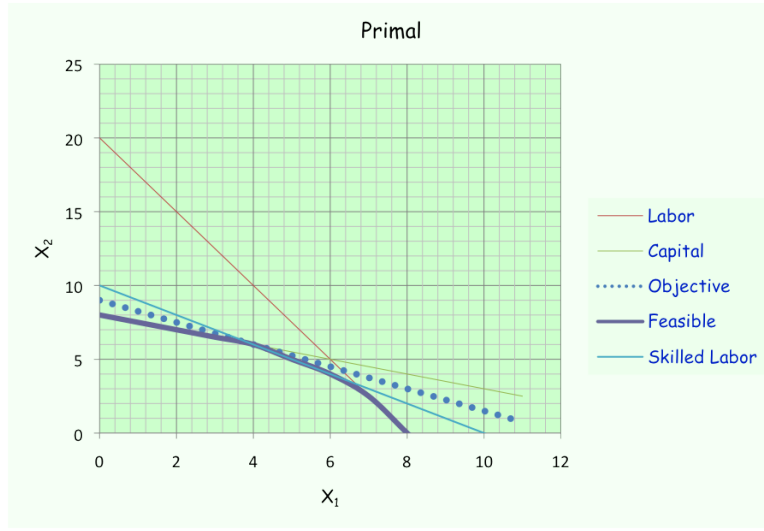


Figure 1: Solution to the primal

with the augmented vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

where s refers to slack and b is unchanged. We then have $Ax = b$ as a system of 3 and 6 unknowns. There are an infinite number of solutions. The *basis theorem* states that for a system of m equations and n variables, where $n > m$, a solution in which $n - m$ variables equal zero is an extreme point. Thus, by setting $n - m$ variables equal to zero and solving the m equations for the remaining variables, an extreme point or basic solution can be found. The number of basic solutions, N_{bs} is given by

$$N_{bs} = \frac{n!}{m!(n - m)!}$$

so this could be a big job! Fortunately LINDO and Excel using solver can handle these larger problems.

13. The Dual: In the primal of this example, only two constraints are binding, skilled labor and capital. By complementary slackness, the dual variable, or *shadow value*, of unskilled labor should then be zero. We cannot easily graph a three dimensional problem, but by the basis theorem, we can solve for the shadow values of skilled labor and capital.

$$[y_2, y_3] \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

The solution is $y = p\bar{A}^{-1} = [0.667, 1]$. Here \bar{A} is a square matrix (must be to have an inverse) that is constructed from the matrix A above by deleting the first row corresponding to unskilled labor.