Matrices, vectors and scalars

Consider the following examples

A **matrix**: is rectangular array of numbers with n rows and m columns.

Example a. $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$ c. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

- The first is square with $n = 2$ and $m = 2$, the second is rectangular, $n = 2$ and $m = 3$. The third is written with its row-column index $a_{row,col}$ explicit so that a_{ij} is the *element* in the *ith* row and *jth* column.
- The only restriction on n and m is that the be integers, i.e., whole numbers greater than or equal to one. There are some important special cases–discussed in the margin notes.

A **scalar** is matrix with n and m both equal to one. If $n = 1$ but m is greater than one 1, the matrix is called a **row vector** and is written $X = \{x_j\}$ with j as the column index. If $m = 1$ but n is greater than one 1, the matrix is called a **column vector** and is written $X = \{x_i\}$ with i as the row index. A **transpose** of a matrix reverses its row column index

$$M^T = \{m_{ji}\}$$

the transpose of a row vector is a column vector and vice-versa.

Operations

Scalar multiplication The product of a scalar z *and a matrix* $A = \{a_{ij}\}$ is $zA = B\{za_{ij}\}$

Note the convenient way we have represented the matrix A as the row-column index on the same, but lowercase letter, a_{ij}. The matrix B is the product and is the same size a A was before it was multiplied.

Example Let $z = 3;$ and $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$. We then have $zA = B$ where $B = \begin{bmatrix} 6 & 3 \\ 0 & 9 \end{bmatrix}$

Matrix multiplication

The product of two matrices, A and B is equal to C under some restrictive conditions. First A and B must "**conform**" to multiplication. This means that column index of A must be equal to the row index of B. The dimension of C is then the row index of A and the column index of B. Second, the multiplication must follow a specific pattern. To get the first element of C, that is c_{11} take the first row of A and multiply it element-by-element times the first column of B and the add up the results. To get the get the second element of C, that is, c_{12}, take the first row of A and multiply it by the second column of B, element by element, add up the results. The element c_{ij} is then

$$c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$$

Example The product of the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\begin{bmatrix} 5 & 8 \\ 9 & 12 \end{bmatrix}$.

The matrices **conform** since both are 2×2 and thus the column index of the first matrix is equal to the row index of the second. The product matrix has the row index of the first and the column index of the second.

Example $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix}$

Example $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -2 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -12 & 12 \\ 14 & -6 \end{bmatrix}$

Properties of Matrix Multiplication

1. Matrix multiplication is finicky about the *order* of multiplication. While we normally think that $ab = ba$, it is *not* true for matrices (except for some special cases). If we reverse the order of multiplication above, then we get $\begin{bmatrix} 2 & 7 \\ 6 & 15 \end{bmatrix}$, an entirely different matrix. We use the terms *postmultiply* to indicate multiplication from the right and *premultiply* to indicate multiplication from the left.

2. It is possible to multiply rectangular matrices as long as they conform and the result if another rectangular matrix. In the example above, matrices a and b do conform because a is a 2×2 and b is a 2×3. The product matrix c is a 2×3. Reversing the order of matrix multiplication in this case does not work because they do not conform; the column index of b is 3 and this does not agree with the row index of a.

3. Two row vectors cannot be multiplied times each other, nor can two column vectors. A special case of matrix multiplication is a row vector multiplied times a column vector. Since the row vector has a row index of one and a the column has column index of one the product is a scalar or 1×1 matrix. This multiplication is called a *scalar, inner or dot product.*

Identity Matrix

The $n \times n$ *identity* matrix I has ones down the main diagonal (upper-left corner to lower-right corner) and zeroes elsewhere. The 3×3

identity matrix, for example, is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 1: Wassily Leontief, 1906-1999, is regarded as the father of input-output analysis. Input-output was partly inspired by Quesnay's Tableau Economique and has been a mainstay of economics and economic policy and planning throughout the world for the past half-century.Formal input-output analysis developed at Harvard after 1932 to begin constructing an empirical example of his input-output system–an effort that gave rise to his 1941 classic, *Structure of American Industry*. Leontief won the Nobel memorial prize in 1973.

The Leontief Inverse

The *inverse* of an $n \times n$ matrix A is an $n \times n$ matrix B satisfying $AB = I$.

1. Input-output matrix A has a Leontief inverse

$$(I - A)^{-1}$$

that allows us to solve the *basic input-output equation*

$$X = AX + F$$

where X is **gross value of production** and includes intermediate goods, AX. There is double counting in the gross value of production and therefore it is not equal to GDP. In the case of a two sector economy, with

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where the input-output coefficient is $a_{i,j}$ is the amount of *input i* required for one unit of *output j*. The product AX is then

$$AX = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} a_{11}X_1 + a_{12}X_2 \\ a_{21}X_1 + a_{22}X_2 \end{bmatrix} \quad (1)$$

Note that the two quantities on the right-hand side are *amounts of agriculture and industry used as intermediate inputs*. Convince yourself that this is indeed the case.

Solving the model

The **parameters** of the input-output models are

(a) all the a_{ij}s. The are assumed to be known, fixed and given.

(b) the vector F of final demands.

The **variables** of the model are the Xs, in this case 2 of them. The vector equation

$$X = AX + F$$

above is essentially *two equations* and therefore we have a complete model, with two equation and two variables, solved for in terms of the 6 parameters of the model. This is done by finding a matrix inverse. First move the AX from the right to the left-hand side of the equation.

$$X - AX = F$$

Next: factor out X

$$(I - A)X = F$$

Note that X *post multiplies* A so it must be factored out on the right of the parens. Next *premultiply by*

$$(I - A)^{-1}(I - A)X = (I - A)^{-1}F$$

where the term on the left is a matrix times its inverse, which must give the identity matrix multiplied by X, which must give X itself. The solution is then

$$X = (I - A)^{-1}F \qquad (2)$$

The question is what to do about the inverse? There are procedures for getting an inverse matrix that are covered in courses on linear algebra. Here we are going to take a short-cut, exploiting the specific nature of the so-called Leontief inverse. We propose that the inverse is given by

$$(I - A)^{-1} = I + A + A^2 + ... + A^n$$

where n needs not be much larger than 5 for a "good" approximation.

2. To see that this indirect method of computing an inverse works, note that

$$(I - A)(I - A)^{-1} = I$$

should be just the identity matrix, so that

$$\begin{aligned}(1 - A)(I - A)^{-1} &= (I - A)(I + A + A^2 + ... + A^n)\\ &= (I + A + A^2 + ... + A^n) - (A + A^2 + ... + A^{n+1})\\ &= I - A^{n+1}\end{aligned}$$

so that if $\lim_{n\to\infty} A^{n+1} = 0$, the approximation is valid.

3. For $\lim_{n\to\infty} A^{n+1} = 0$ the matrix A must be *productive.* A productive matrix is one for which the AX vector is inside the space delimited by the X vector, as seen below.

This is a nice approximation since it has an economic interpretation. From equation 2 we need only post-multiply equation 1 by F

$$X = (I - A)^{-1}F = F + AF + A^2F + ... + A^nF \quad (3)$$

This equation says that in order to determine how much X to produce, one must first produce F and then the inputs to produce F, which are AF. Then one must produce the inputs to produce the inputs for F, which is A_2F, and so on.

We have solved the input-output model in equation 3 without having to compute the inverse matrix of $(I - A)$. This is the solution to the model.

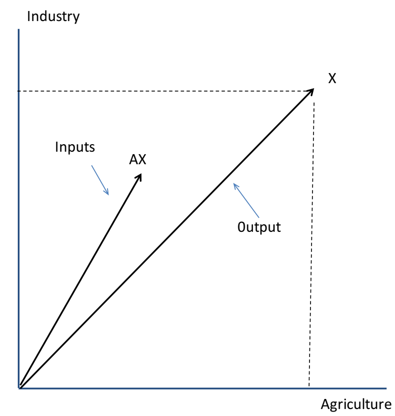

Figure 2: Vector diagram of the input-output model. Note that inputs must be less than or equal to outputs for a closed economy.

Vectors diagrams

By far the most attractive feature of input-output models is their ability to account for economy-wide effects of project implementation. A dam project in a developing economy, for example, is said to "pull" on inputs, both domestic and foreign, which in turn pull on their own suppliers. How this works is easy to see in vector diagrams. Figure 2 shows the output and input vector for a two sector economy, agriculture and industry. This economy is *productive* since its inputs are bounded from above by the dotted lines. If the inputs were greater than this upper bound, some level of *imports* would be indicated.

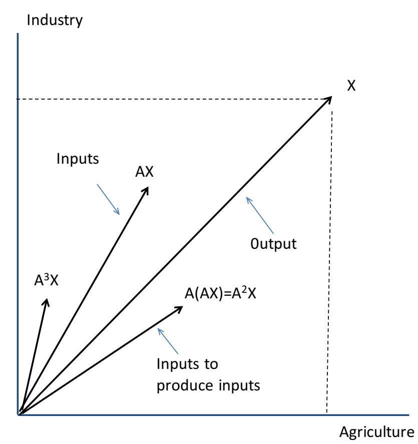

Figure 3: Vector diagram of the input-output model. Note the shrinking size of the input vectors as they chain.

Figure 2 shows the essence of the input-output model. As inputs are required for the outputs, so too are inputs required for the inputs themselves. This is shown in the figure as the even shorter vector labeled A^2. The process continues *ad infinitum* but practically only to the point that the level of inputs is indistinguishable from zero. The sum of these vanishing vectors is known as the *direct and indirect requirements* for the production of X. This is a well-known concept and shows that it is not possible in modern economies to produce anything without effectively rippling through the *entire* economy, for both output and employment.

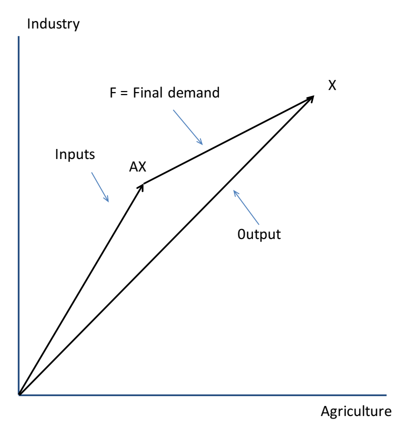

Figure 4: Vector diagram of final demand in an input-output model. Note that F is just the difference between outputs and inputs in the model.

Note that F is shown in figure 4 as the difference between X and AX. Here F is a compact way of writing the more familiar national income and product accounting equation

$$F = C + I + G + N_x$$

where C is consumption (including imports), I is the sum of structures, equipment, residential construction and change in inventories undertaken by both private and public sectors, G is current government consumption and N_x is net exports. When the dam is built it will enter final demand as investment (assuming it takes less than or equal to one year). Equation 3 shows how to determine X.

Employment in input-output models

With data in hand on the wage bill for the dam project (subject to the caveats mentioned above about its composition) one can perform an employment impact assessment of the project. Without the aid of the conceptual framework developed so far, one might be tempted to write the employment total, $\mathcal{L}$, as

$$\mathcal{L} = LF = \sum_{i=1}^{n} l_i F_i$$

where $L = [l_1, l_2, ..., l_n]$ is a row-vector of labor coefficients or ratios of the wage bill to the level of outputs of each sector. This would, however, be incorrect since it omits the employment generated by the production of the inputs for F. It also omits the inputs to produce the inputs and so on. The correct expression for total employment is

$$\mathcal{L} = LX = \sum_{i=1}^{n} l_i X_i > \sum_{i=1}^{n} l_i F_i.$$

The employment *impact* in the case of the dam, $\Delta\mathcal{L}_d$, can be written as

$$\Delta\mathcal{L}_d = \begin{bmatrix} l_1 l_2 \end{bmatrix} \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^2 + \cdots + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^n \right] \begin{bmatrix} 0 \\ \Delta F_2 \end{bmatrix}$$

The direct employment, $\Delta\mathcal{L}_{d'}$, is

$$\Delta\mathcal{L}_{d'} = \begin{bmatrix} l_1 l_2 \end{bmatrix} \begin{bmatrix} 0 \\ \Delta F_2 \end{bmatrix}$$

whereas the direct plus indirect employment is $\Delta\mathcal{L}_d$.

Exercises

Example 1. Determine the total demand for industries 1,2 and 3 given the matrix of technical coefficients and final demand vector F.

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.5 & 0.2 \end{bmatrix}, F = \begin{bmatrix} 150 \\ 200 \\ 210 \end{bmatrix}$$

Solution: First form $(I - A)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.5 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.3 \\ -0.3 & -0.5 & 0.8 \end{bmatrix}$$

And then write the Leontief inverse

$$(I - A)^{-1} = \begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix}$$

The last step is to premultiply the F vector by the Leontief inverse

$$\begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix} \begin{bmatrix} 150 \\ 200 \\ 210 \end{bmatrix} = \begin{bmatrix} 879.5 \\ 1023.9 \\ 1232.2 \end{bmatrix}$$

2. A massive new public works program is initiated. Compute the change in total production if final demand increases by 40, 20 and 25 respectively.

Solution: We have

$$\Delta X = (I - A)^{-1} \Delta F = \begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix} \begin{bmatrix} 40 \\ 20 \\ 25 \end{bmatrix} = \begin{bmatrix} 152.1 \\ 150.6 \\ 182.4 \end{bmatrix}$$

3. Let the direct labor employed per unit of output be given by

$$L = \begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix}$$

Compute the employment impact of the new project.

Solution: We have a change in employment equal to the labor required by the change in final demand; that is:

$$L\Delta X = L(I - A)^{-1} \Delta F. \begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix} \begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix} \begin{bmatrix} 40 \\ 20 \\ 25 \end{bmatrix} = 107.74$$

This amounts to an increase of

$$\begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix} \begin{bmatrix} 879.5 \\ 1023.9 \\ 1232.2 \end{bmatrix} = 107.74/699.15 = 0.154$$

or a 15% increase in employment. The denominator, 699.15 is given by

$$L(1-A)^{-1}F = LX = \begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix} \begin{bmatrix} 879.5 \\ 1023.9 \\ 1232.2 \end{bmatrix} = 699.15$$