Check your understanding: Input-output models

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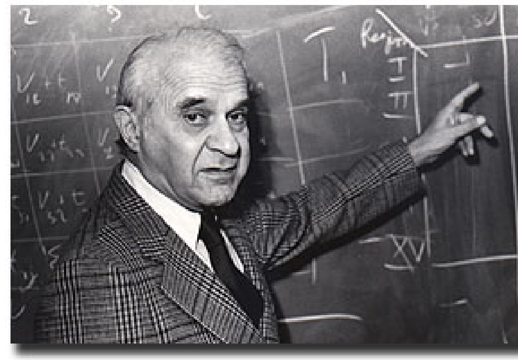
Input-output models

Input-Output and Social Accounting Matrices



Input-output was partly inspired by the Marxian and Walrasian analysis of general equilibrium via interindustry flows. It has been a mainstay of economics and economic policy and planning throughout the world for the past half-century and used for economic planning throughout the developed and developing world.

Leontief was raised in Russia and obtained his Ph.D. in Berlin. He developed input-output analysis at Harvard after 1932 and he bagan constructing an empirical example of his input-output system, his 1941 classic, *Structure of American Industry*. Leontief followed up this work with a series of classical papers on input-output economics.

Simple Example

Let there be two sectors agriculture and industry. Agriculture requires 2 units of industry to produce 10 units of output while industry requires 3 units of agriculture to produce 20 units of output. Each uses its output as an input, perhaps 10 percent of its gross output. The A matrix or input-output matrix in coefficient form

$$\begin{bmatrix} 1/10 & 3/20 \\ 2/10 & 1/10 \end{bmatrix}$$

or

$$\begin{bmatrix} 0.1 & 0.15 \\ 0.2 & 0.1 \end{bmatrix}$$

here the coefficient a_{ij} is the amount of good i used for the production of *one unit of good* j. The *coefficient matrix* must be distinguished from the *flow matrix* $\tilde{A} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

Here X is column vector of gross value of production $X = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

Product of A and X is AX is a quantity of intermediate use of both agriculture and industry as illustrated below.

Final demand

The column vector F is **final demand**, consumption, investment, government and final demand, $F = C + I + G + N_x$ $F = \begin{bmatrix} 8.5 \\ 16 \end{bmatrix}$

Note that intermediate demand is *not* part of GDP and is also not part of final demand. Input-output analysis solves fundamental problem of what level of gross outputs X is required if a specific final demand vector F is to be produced?

The material balance equation

The fundamental problem of input-output analysis is thus to solve the material balance equation

$$X = AX + F \tag{1}$$

which just says that gross output is the intermediates plus final demand. The question is how to solve this equation for X. For this we need linear algebra since the material balance is only system of simultaneous linear equations.[1]

[1] see CYU on linear algebra for full explanation.

At each stage of production some intermediate goods, raw materials and other inputs, are required. These are similar to capital goods, but in fact only last a fraction of the period of production while capital goods last longer than one period. Let X_j be the output of one of n sectors in a given period. A simple average cost function would be

$$p_j X_j = \sum_{i}^{n} p_i a_{ij} X_j + \sum_{l=1}^{m} w_{lj} l_{lj} + r_j k_j \tag{2}$$

where p_i is the price of good i, with $i, j = 1, 2, ..., n$, w_j is the wage paid, l_{lj} is the amount of labor of skill class $l = 1, 2, ..., m$ and r_j is the cost of capital. The amount of capital required per unit of output is k_j. The key parameter for input-output (I-O) analysis is a_{ij}, the amount of good i required as an intermediate or raw material for the production of one unit of good j.

Capital and labor coefficients are usually considered to be variable over time and across sectors. The coefficients depend on the wage-rental ratio, w/r, for each skill category of labor. Given that ratio, firms choose a combination of capital and labor that minimizes the cost to produce the level of output X_j. The key assumption for I-O analysis is that, while factor proportions vary with the wage-rental ratio, the a_{ij} proportions of the intermediate and raw material inputs *do not*.

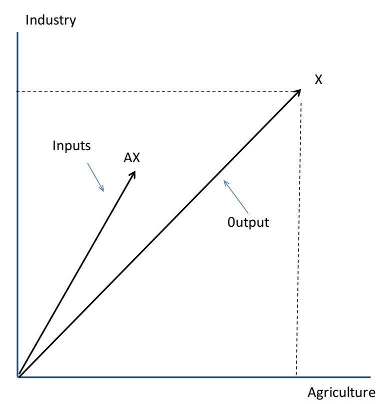
By far the most attractive feature of I-O models is their ability to account for economy-wide effects of project implementation. An infrastructure project is said to "pull" on inputs, both domestic and foreign, which in turn pull on their own suppliers. To see this let the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{3}$$

be the I-O *coefficient* matrix describing the amount of good i required for one unit of the production of good j, as noted above. These are the *flow* coefficients and have embedded in them the fixed-coefficient assumption already discussed. The total demand for intermediate goods is an $n \times 1$ matrix, or column vector.

$$AX = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \tag{4}$$

Figure 1: Vector diagram of the I-O model-âĂŞinputs must be $\leq$ outputs for a closed economy



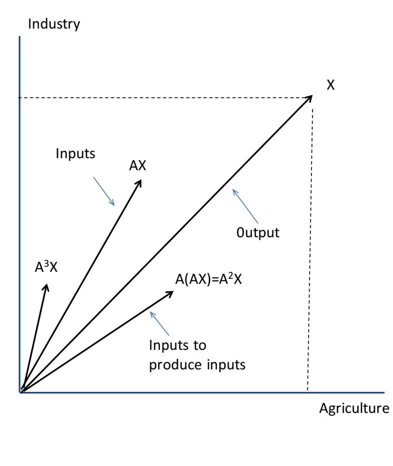
which is the same as $\sum_{j=1}^{n} a_{ij} X_j$. Figure 1 shows the output and input vectors for a two-sector economy, agriculture and industry. This economy is *productive* since its inputs are bounded from above by the dotted lines. If the inputs were greater than either or both upper bounds, some level of *imports* would be indicated.

Figure 2 shows the essence of the I-O model. As inputs are required for the outputs, so too are inputs required for the inputs themselves. This is shown in the figure as the even shorter vector labeled A^2. The process continues ad infinitum but practically only to the point that the level of inputs for that round is indistinguishable from zero. The sum of these vanishing vectors is known as the *direct and indirect requirements* for the production of X. This beloved concept among practitioners of I-O analysis shows that it is not possible in modern economies to produce anything without effectively rippling through the *entire* economy, for both output and employment.

How then is X itself determined? It can be deduced from what is known as the *material balance equation* of the I-O framework. Implicitly define the vector of *final demand*, F as

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} \tag{5}$$

Figure 2: Vector diagram of the I-O model–note the shrinking size of the input vectors as they chain and sum to the direct and indirect input requirements for the given value of X



This expression is an expanded version of the material balance equation above, repeated here for convenience as

$$X = AX + F$$

where F is shown in figure 3 as the *difference* between X and AX. There are many ways to solve this last expression for X as a function of final demand, F. In the early days of the Soviet Union they did it with buildings full of *computers*, literally women armed with pencils and paper, but now it can be done by matrix inversion methods available even in Excel. A more instructive solution is the so-called power-series expansion or approximation to the solution that involves successive powers of A multiplied by the final demand vector F

$$X \sim F + AF + A^2F + ... + A^nF \tag{6}$$

which provides the basis for quick computation of X given F in virtually any programming language.[2] In words, this expression says: to produce F one needs also to produce inputs for F and then inputs to produce the inputs for F and so on. The gross output, X, is then the sum of these inputs plus the quantity of F itself.

[2] The classic reference, **?**, is still the best since it shows the close connection between economic theory and the I-O framework.

To see how an infrastructure project that might treat one or more rivers that deliver phosphorous to Lake Champlain consider the following experiment. Raise the *second* coefficient in the vector F by an amount ΔF_2, the planned expenditure during the period[3]

[3] A numerical example of precisely this infrastructure shock is given below.

$$\begin{bmatrix} F_1 \\ F_2 + \Delta F_2 \end{bmatrix}$$

This says the project *direct* inputs will be drawn from the industrial sector. Indirect inputs will, of course, be drawn from all sectors. Recall that F is a compact way of writing the more familiar national income and product accounting equation

$$F = C + I + G + N_x$$

where C is consumption (including imports), I is the sum of structures, equipment, residential construction and change in inventories undertaken by both private and public sectors, G is current government consumption and N_x is net exports, exports minus imports. When the infrastructure is built, it will enter final demand as investment (assuming it takes less than or equal to one year).[4] It will then begin to pull on its intermediate inputs, which in turn pulls on theirs in an infinite but asymptotic chain.

[4] If not, the stimulus is spread over the entire project period as seen in the example below.

Employment in I-O models

With data in hand on the wage bill for the infrastructure project (subject to the caveats mentioned above about its composition) one can perform an employment impact assessment of the project.[5] Without the aid of the conceptual framework developed so far, one might be tempted to write the employment total, $\mathcal{L}$, as

$$\mathcal{L} = LF = \sum_{i=1}^{n} l_i F_i$$

where $L = [l_1, l_2, ..., l_n]$ is a row-vector of labor coefficients or ratios of the employment to the level of gross outputs of each sector, derived from the base SAM in which wages and prices are conventionally set to one. This would, however, be incorrect since it omits the employment generated by the production of the inputs for F. It also omits the inputs to produce the inputs and so on, as just discussed. The correct expression for total employment is

$$\mathcal{L} = LX = \sum_{i=1}^{n} l_i X_i > \sum_{i=1}^{n} l_i F_i.$$

The employment *impact* in the case of the infrastructure, $\Delta\mathcal{L}_d$, can be written as

$$\Delta\mathcal{L}_d = \begin{bmatrix} l_1 l_2 \end{bmatrix} \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^2 + \cdots + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^n \right] \begin{bmatrix} 0 \\ \Delta F_2 \end{bmatrix}$$

The change in *direct* employment due to the infrastructure project is $\Delta\mathcal{L}_{d'}$, is

$$\Delta\mathcal{L}_{d'} = \begin{bmatrix} l_1 l_2 \end{bmatrix} \begin{bmatrix} 0 \\ \Delta F_2 \end{bmatrix}$$

whereas the change in direct plus indirect employment is $\Delta\mathcal{L}_d$.

Closed and open I-O models

Closed and open I-O models are usefully distinguished (**?**), primarily because the former reveal the inner workings of the multiplier. The model above is open but a closed model would include coefficients for consumption as well as intermediates. These are known as *labor-feeding coefficients*, c_{ij},

$$c = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}$$

which measure the amount of good i labor must *consume* to produce one unit of *labor*. In some sense this is an odd concept, but again

[5] The concepts explained in this section will apply to Social Accounting Matrices, SAMs, both static and dynamic, that are discussed in the next sections.

the coarse-grain nature of I-O modeling allows for a large amount of detail to be built in, ex post. Total consumption demand for this economy is

$$\mathcal{C} = cLX$$

Here equation 5 above, the material balance equation, can be altered to include the labor-feeding coefficients by replacing A with

$$A^{+} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} + \begin{bmatrix} c_{11}l_1 & c_{12}l_2 & ... & c_{1n}l_n \\ c_{21}l_1 & c_{22}l_2 & ... & c_{2n}l_n \\ \vdots & \vdots & & \vdots \\ c_{n1}l_1 & c_{n2}l_2 & ... & c_{nn}l_n \end{bmatrix}$$

which can be compactly written as

$$X = A^{+}X + F$$

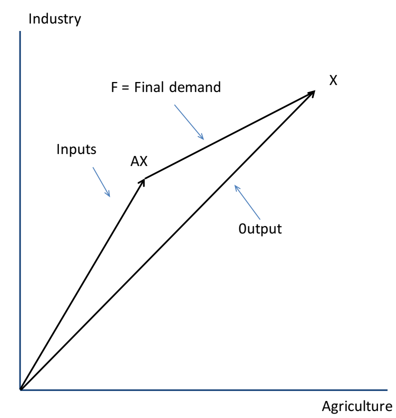
where $A^{+} = A + CL$. The system can then be solved for a given level of final demand, F.

$$X \sim F + A^{+}F + (A^{+})^2F + ... + (A^{+})^nF$$

The rest of the employment analysis follows in the wake of this rather fundamental change in the nature of the model. The labor-feeding coefficients, c_{il}, are not easy to estimate and embody an unrealistic assumption, namely, that workers do not adjust their consumption bundle in response to changes in the macro economy, whether incomes or prices. So far, of course, there are no relative prices in the I-O model and so one could presumably sleep soundly, even after making this assumption. More realistically, side calculations could be made by hand on the time-path of the c_{ij} coefficients as relative prices do in fact evolve.

What workers require to live and keep working admittedly suggests an excessively rigid vision of the economy, but there is one attractive feature of the closed model that could in principle be used to establish an upper bound for the impact of an investment project shock. Figure 4 shows that F, A and $A^{+}X$ are all collinear and this corresponds to the maximal rate of growth of the closed system and therefore the maximal rate of growth of employment.[6] Employment in the economy is $\mathcal{L} = LX$.

Figure 3: Vector diagram of final demand in an I-O model



[6] The eigenvector associated with the maximal eigenvalue gives the proportions for balanced growth along this von Neumann turnpike. The surplus over and above the technical and labor-feeding coefficients, $S = (I - A^{+})X$ is itself a vector of goods that may be consumed, invested or exported. Only if *all* the surplus is reinvested will a closed economy grow at its fastest sustainable rate.

Prices and dual of the material balance

Since the I-O model is a matrix, it can be viewed either as a “stack" of rows or a “rack" of columns. Viewing the problem from a row perspective gives a material balance, a balance between supply and demand. The price level for each row is given by

$$PX = PAX + PC + PI + PE - eP^{*}M \tag{7}$$

where $P = [p_1, p_2, ..., p_n]$, A is the I-O coefficient matrix, $C = [C_1, C_2, ..., C_n]'$ is the consumption column vector, $I = [I_1, I_2, ..., I_n]'$ is the level of investment[7], $E = [E_1, E_2, ..., E_n]'$ is exports and $M = [M_1, M_2, ..., M_n]'$ is the level of imports. The nominal exchange rate is given by e and the price of imports is P^{*}. Here PAX is intermediate demand and $PC + PI + PE - eP^{*}M$ is final demand. Factor demand for labor is denoted $L = [L_1, L_2, ..., L_n]$ and capital is given by $K = [K_1, K_2, ..., K_n]$. Value added, V, is then

$$V = PX - PAX = wL + rK$$

where w is the wage rate and r is the rate of return to capital. Value added must equal the value of final demand since

$$PF = PX - PAX$$

or $V = PF$. It should be clear from this discussion that I-O is less of an economic model than it is a framework to record the *data* of an economy; in other words, a highly structured data-base.

Columns of the matrix require prices since, reading down the columns of the I-O matrix, the goods are heterogeneous and must be aggregated by the price vector. To this nominal value is added the return to labor, wL, and the return to capital, rK, which are both measured in nominal terms. This suggests that the entire presentation of the I-O framework must be in *nominal* terms. If a given I-O system is also compiled for a *base year*, the values in the structured data base are both real *and* nominal. As noted above, it is convenient to normalize all prices to one for the base year, along with the base year wage rate. The units of X are then in millions of LCUs and if prices were to rise by, for example, 15 per cent, it could be said that PX is the *cost of what could have been purchased in the base year* with one-million LCUs of the base year. This is a useful convention in I-O accounting and widely adopted.

In table 1 there are two sectors, industry and agriculture. Here the GDP, computed as the sum of value added in both sectors, is 48+70 = 118. Assume, unrealistically, that the infrastructure can be

Figure 4: Vector diagram of balanced growth with maximal employment growth

/Users/wgibson/ILO/graphics/IO_4.pdf

[7] The apostrophe indicates vector transpose.

Table 1: I-O framework

	Agriculture	Industry	Consumption	Investment	Government	Exports	Total
Agriculture	10	8	30	5	15	10	**78**
Industry	20	12	40	8	15	-5	**90**
Value Added	48	70	-	-	-	-	-
labor	30	42	-	-	-	-	-
Capital	18	28	-	-	-	-	-
Total	**78**	**90**	**70**	**13**	**30**	**5**	-

- = N/A

Millions of LCUs.

Source: Authors' computations based on illustrative data.

Table 2: An infrastructure project

	Agriculture	Industry	Cons	Investment	Govt	Exports	Total
Agriculture	102	8.12	30	5	15	10	**78.1**
Industry	204	12.19	40	9.18	15	-5	**91.4**
Value added	48.09	71.09					
Labor	306	42.66					
Capital	18.03	28.44					
Total	**78.1**	**91.4**	**70**	**14.2**	**30**	**5**	-

Millions of LCUs.

Source: Authors' computations based on illustrative data.

constructed in one year and that its construction requires an expenditure of one per cent of GDP, the minimum the framework is capable of seeing. This is 1.18 million LCUs and is added to the level of investment demand for industry.[8] In the I-O framework this rise in aggregate demand gives direct and indirect effects on the both sectors as seen in table 2.

From table 2 it is evident that investment has risen from 8 to 9.18, or one per cent of GDP. This change causes direct and indirect effects on both industry *and* agriculture precisely in the fashion described in the mathematical model above. This is the simplest possible model inasmuch as consumption remains fixed and only intermediate demand rises in response to the infrastructure shock. In particular, the capital and labor components of total value added rise proportion-

[8] Keep in mind that the investment above is investment by *origin* and not destination. The former is a component of aggregate demand whereas the latter is an increment in the capital stock.

ately. The demand for labor increases according to labor coefficients calculated from the base (30/78 for agriculture and 42/90 for industry). The same occurs with capital in this entirely linear model.

The example identifies the limitations of the linear I-O model as a framework for analyzing the impact of infrastructures on the demand for labor. It is unrealistic to think that the cost of capital will rise proportionately with output in the short run. It might be better to have the capital fixed in the short run so that the cost of capital itself remains fixed. In the short run, output increases in both sectors, but employing more workers with the same amount of capital is satisfied by the rise in demand.

How does the demand for labor actually change when an infrastructure is introduced? In the short run, capital is fixed, as is the nominal wage rate. The assumption is that the project managers will employ labor until the value of its marginal product is just equal to the real wage. Any other assumption would violate the most basic principles of profit maximization, implicitly allowing firms to employ more or less labor than they need.

The literature on project analysis suggests that “satellite" models can be constructed to evaluate the impact of project investment on the economy, they mean that one of the most crucial assumptions for the construction of a satellite model is that there should be a “representative firm" for the satellite that shows how it reacts to the infrastructure shock. This will allow for a more realistic non-linear response to the increase in demand. The satellite model must be consistent with the value added in the I-O model.

Assume first that the industrial sector production function is given by

$$p_j x_j - \sum_{i=1}^{n} p_i a_{ij} x_j = \mathcal{A}_j K_j^{\beta_j} L_j^{1-\beta_j} \tag{8}$$

where $\mathcal{A}_j$ is an arbitrary calibration constant that can be used to model technological change or spending on environmental protection. Here K_j is the capital employed by the industrial firm and L_j is the labor. Nominal value added, v_j is given by

$$v_j = p_j x_j - \sum_{i=1}^{2} p_i a_{ij} x_j$$

Since this is a Cobb-Douglas production function, the marginal products for capital and labor, respectively are

$$\begin{aligned} \beta_j v_j / K_j &= r \\ (1-\beta_j) v_j / L_j &= w \end{aligned} \tag{9}$$

where for the moment, w and r are common to all sectors. Under the standard neo-classical assumption that a rational firm would employ labor until the value of the marginal product of labor is equal to the wage and the value of the marginal product of capital is equal to the cost of capital, these two equations can be rearranged to yield

$$\beta_j = rK_j/v_j$$
$$(1-\beta_j) = wL_j/v_j$$

The right-hand side in both cases is the share of the factor in the value of total production, precisely what is shown in the last two rows of the I-O matrix in table 1. In this case

$$\beta_1 = 18/(18+30) = 0.375$$
$$\beta_2 = 28/(28+42) = 0.4$$

Once the share of capital, β_j is estimated, the share of labor can be deduced from the assumption of constant returns to scale (that the exponents in the production function add to one). The labor shares are

$$1-\beta_1 = 0.625$$
$$1-\beta_2 = 0.6$$

where it is seen that the factor shares are similar between the two branches of production.

Why is the Cobb-Douglas production function used here? Observe that the "adding-up" feature of the Cobb-Douglas is fully consistent with the I-O framework and can be easily calibrated from data of the matrix. Here the structure of the data determines the production function. There is no real reason why the exponents would necessarily add to one, but if the Cobb-Douglas is to be consistent with the I-O structure, it must.

This is one of many examples in which the *structure* rather than the data determines, *or limits*, what can be said using a model. One of the reasons that I-O is so popular in the developing world is that it imposes strict constraints on what the data can say. This can be seen as both a benefit and a cost of the method.

If the percentage is given in the short-run, the year for which the data is collected, but labor is variable and determined by the marginal productivity condition, the firm is on its rising marginal cost curve. Any increase in output then must be accompanied by a rise in demand for labor. As output x_i rises so too does v_i. Equation 9 shows that the marginal product of labor and capital rise proportionately. In the case of labor, employment rises and the wage rate is fixed. For capital, however, the quantity is fixed and thus the rate of return rises above the market rate. The assumption of the short run drives this result but as capital invested changes in the next period, the model is in a new "short-run" for which the same logic applies.

Equation shows how the new values of X can be computed. The base level is given by

$$\begin{bmatrix} 78 \\ 90 \end{bmatrix} \sim \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.128 & 089 \\ 0.256 & 0.133 \end{bmatrix} + \begin{bmatrix} 039 & 023 \\ 067 & 041 \end{bmatrix} + \begin{bmatrix} 01 & 01 \\ 02 & 01 \end{bmatrix} + \begin{bmatrix} 003 & 002 \\ 005 & 003 \end{bmatrix} \right] \begin{bmatrix} 60 \\ 58 \end{bmatrix}$$

where this calculation, easily done in Excel, is taken only to the 4th power for ease of exposition.[9] The rule of thumb for a minimum shock is approximately one per cent of GDP, or 1.18, that would be added to the second component in the F vector to represent the increase in final demand for the infrastructure project.[10] This gives

$$X = \begin{bmatrix} 78.1 \\ 91.4 \end{bmatrix}$$

assuming no change in the price level. The new value added is calculated as the difference between the value of output and intermediate costs at the new level of X, as seen in table 2

$$v_1 = 48.1$$
$$v_2 = 71.1$$

then all the increase in value added must appear as added demand in the labor market and a rise in the rate of return to capital. The increase in employment is calculated as $\Delta L = l_i(x_i - x_{i0})$ where x_{i0} is the base level of output.

$$\Delta L_1 = 06$$
$$\Delta L_2 = 0.66$$

Note that the rate of return to capital also increases slightly, assuming that the capital stock remains fixed.[11]

[9] The error is less than 0.2 per cent. Any degree of accuracy could be obtained by extending the power series further.

[10] The model does not "see" small shocks but is just as blind to large shocks that induce sufficient structural change to do violence to the assumed stability of the parameters of production and consumption.

[11] The calculation is tedious but straightforward. First compute K from the Cobb-Douglas equation 8 taking the wage rate as 1 and L from the base matrix. The shares β and $1 - \beta$ are also taken from the base matrix. The result is $K_1 = 105.1$ and $K_2 = 150.6$. The rate of return to capital–or the cost of a unit of capital–can be then calculated for each sector from the fact that rK can be read from the base matrix. This gives $r_1 = 0.17$ and $r_2 = 0.186$. Holding K constant during the shock produces a change in r of 0.172-0.171 = 0.001 and 0.189-0.186 = 0.003 for the two sectors respectively.

Exercises

1. Consider the data

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.5 & 0.2 \end{bmatrix}$$

with

$$F = \begin{bmatrix} 150 \\ 200 \\ 210 \end{bmatrix}$$

Determine the total demand for industries 1,2 and 3 given the matrix of technical coefficients and final demand vector F.

Solution: First form

$$(I - A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.5 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.3 & -0.2 \\ -0.4 & 0.9 & -0.3 \\ -0.3 & -0.5 & 0.8 \end{bmatrix}$$

Next form the adjoint matrix

$$\begin{bmatrix} 0.57 & 0.34 & 0.27 \\ 0.41 & 0.58 & 0.32 \\ 0.47 & 0.49 & 0.6 \end{bmatrix}$$

Finally, divide by the determinant 0.239 *to* get the Leontief inverse

$$(I - A)^{-1} = \frac{1}{0.239} \begin{bmatrix} 0.57 & 0.34 & 0.27 \\ 0.41 & 0.58 & 0.32 \\ 0.47 & 0.49 & 0.6 \end{bmatrix} = \begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix}$$

The last step is to premultiply the F vector by the Leontief inverse

$$\begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix} \begin{bmatrix} 150 \\ 200 \\ 210 \end{bmatrix} = \begin{bmatrix} 879.5 \\ 1023.9 \\ 1232.2 \end{bmatrix}$$

2. A massive new public works program is initiated. Compute the change in total production if final demand increases by 40,20 and 25 respectively.

   Solution: We have $\Delta X = (I - A)^{-1} \Delta F$.

$$\begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix} \begin{bmatrix} 40 \\ 20 \\ 25 \end{bmatrix} = \begin{bmatrix} 152.09 \\ 150.63 \\ 182.43 \end{bmatrix}$$

3. Let the direct labor employed per unit of output be given by

$$L = \begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix}$$

Compute the employment impact of the new project.

Solution: The change in employment equal to the labor required by the change in final demand; that is: $L\Delta X = L(I - A)^{-1}\Delta F$.

$$\begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix} \begin{bmatrix} 2.3849 & 1.4226 & 1.1297 \\ 1.7155 & 2.4268 & 1.3389 \\ 1.9665 & 2.0502 & 2.5105 \end{bmatrix} \begin{bmatrix} 40 \\ 20 \\ 25 \end{bmatrix} = 107.74.$$

This amounts to an increase of

$$\begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix} \begin{bmatrix} 879.5 \\ 1023.9 \\ 1232.2 \end{bmatrix}$$

$= 107.74$ or $107.74/699.15 = 0.1541$ or a 15% increase in employment.

The denominator, 699.15 is given by $L(1 - A)^{-1}F = LX =$ or

$$\begin{bmatrix} 0.2 & 0.15 & 0.3 \end{bmatrix} \begin{bmatrix} 879.5 \\ 1023.9 \\ 1232.2 \end{bmatrix}$$

$= 699.15$