Check your understanding: Crusoe's problem

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A solution is presented for the slope of an indifference curve and the demand for two goods, fish and cocos, when Robinson Crusoe works a given amount of time and chooses rationally.

Crusoe is shipwrecked on an island with only his labor at his disposal. He needs to consume food to survive. He finds that it takes l_x hours to fell one coconut and l_y hours to catch one fish. He is willing to work $\bar{L}$ hours per period. A basic question asked in economics is how much fish and how many cocos does Crusoe consume if he likes them according to the utility function, U

$$U = \sqrt{xy} \tag{1}$$

where x is the quantity of fish consumed and y in the number of cocos

We use a *mathematical model*: defined as set of equations such that number of variables is equal to the number of equations. Each symbol in the model is either parameter or a variable. We use the model to solve for the variables in terms of the parameters. In this *model of resource allocation* we have the following division between parameters and variables.

- Parameters
  1. *endowment* of resource labor is $\bar{L}$
  2. *technology (labor coefficient*-hours of labor per unit of output) l_y, l_x
  3. *tastes characterized by the utility function* in equation 1 above.
- Symbols $(x, y, U, l_x, l_y, \bar{L})$,
- Variables: (x, y, U)
- Production possibility frontier (PPF)[1]

$$\bar{L} = l_x x + l_y y$$

- Tangency condition: Opportunity cost of cocos in terms of fish in terms of cocos is $p = \frac{l_x}{l_y}$ is the *negative* of the slope of the PPF

$$p = l_x / l_y$$

- Utility function

$$U = \sqrt{xy}$$

[1] It is easy to see in this case that the PPF is just Crusoe's budget constraint.

Recall that we need not solve all three equations for the 3 unknowns. Solve first for x and y and solve for U.[2]

[2] Mathematically, this problem is therefore *decomposable*.

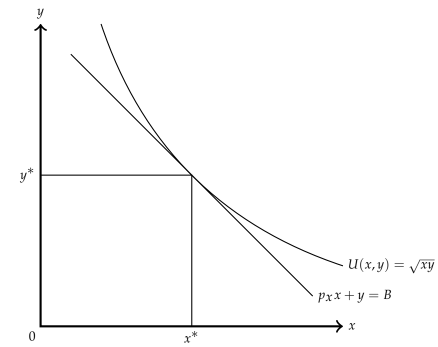


Figure 1: The solution to the consumer's problem for Crusoe

- To find the maximum level of utility, we must set the slope of the indifference curve equal to the slope of the budget constraint, that is, the PPF.
- The slope of the indifference curve is equal to the rise over the run. To make things easy, square both sides of the utility function:

$$U^2 = xy$$

- The indifference curve is defined as having the same level of utility. Thus on *two neighboring points, 1 and 2* on the same indifference curve, we have:

$$U_1^2 = y_1 x_1$$
$$U_2^2 = y_2 x_2$$

Now subtract the two levels of utility to find zero on the left hand side:

$$0 = y_1 x_1 - y_2 x_2$$

Now add and subtract $y_1 x_2$

$$\begin{aligned} y_1 x_1 - y_1 x_2 - y_2 x_2 + y_1 x_2 &= 0 \\ y_1(x_1 - x_2) - x_2(y_2 - y_1) &= 0 \end{aligned}$$

Next divide the last equation by $(x_2 - x_1)$ and rearrange

$$y_1 = x_2 \frac{(y_2 - y_1)}{(x_1 - x_2)}$$

and now by x_2

$$\frac{y_1}{x_2} = \frac{(y_2 - y_1)}{(x_1 - x_2)}$$

which is the same as

$$-\frac{y_1}{x_2} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

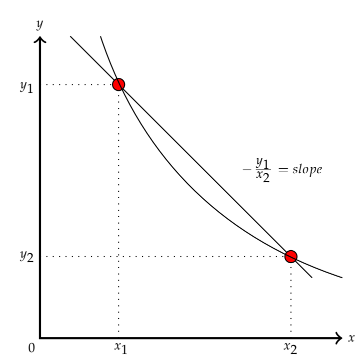
The slope is the *rise over the run*

$$slope = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Thus we have

$$-\frac{y_1}{x_2} = slope$$

of the line joining the two points in figure in the margin.



Now let the two neighboring points come close together, that is let point 2 approach point 1. As the points come together we are no longer taking an average slope of the curve between two points, but the slope at one point. This is called the *instantaneous slope of the indifference curve:*

$$-\frac{y}{x} = \frac{\Delta y}{\Delta x}$$

and is only for the square-root utility function.[3]

[3] For the Cobb-Douglas, $U = x^{\alpha} y^{(1-\alpha)}$, the slope of the indifference curve is

$$\frac{\Delta y}{\Delta x} = -\frac{\alpha}{(1-\alpha)}\frac{y}{x}$$

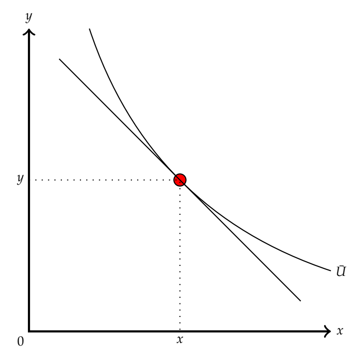


Figure 2: Slope of the indifference curve

- The slope of the indifference curve is called the *marginal rate of substitution* here of fish for cocos. This is how many fish one must substitute to get the same utility when the number of cocos goes down by one unit.
- To find the maximum utility, set the slope of the indifference curve equal to the slope of the budget constraint, or PPF. This is the *tangency condition*
- Thus the marginal rate of substitution of fish for cocos is equal to the opportunity cost of cocos in terms of fish.
- To solve for the optimal combination, use the tangency condition and the PPF and solve simultaneously.

$$\begin{aligned} \frac{l_x}{l_y} &= \frac{y}{x} \\ \bar{L} &= l_x x + l_y y \end{aligned}$$

The solution is[4]

[4] To see that this matches the standard problem in which $x = B/2p, y = B/2$ and $U = B/2\sqrt{1/p}$, note that $B = \bar{L}/l_y$ and $p = l_x/l_y$

$$\begin{aligned} x &= \frac{1}{2}\frac{\bar{L}}{l_x} \\ y &= \frac{1}{2}\frac{\bar{L}}{l_y} \\ U &= \frac{\bar{L}}{2}\sqrt{\frac{1}{l_x l_y}} \end{aligned}$$

Example

Let $l_x = 2$, $l_y = 3$ and $\bar{L} = 12$ Crusoe will consume[5]:

[5] Here $B = \bar{L}/l_y = 4;\ p = 2/3$. So that $x = B/2p = 4/[2(2/3)] = 3$ and $y = B/2 = 4/2 = 2$ and U = $\sqrt{6}$.

$$\begin{aligned} x &= \frac{1}{2}\frac{\bar{L}}{l_x} = 3\ cocos \\ y &= \frac{1}{2}\frac{\bar{L}}{l_y} = 2\ fish \\ U &= = \sqrt{6} \end{aligned}$$