Check your understanding: Utility maximization for consumers

EC 11

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A simplified discussion of the consumer's problem.

Consumer's problem

The consumer's problem is to maximize utility, $U = x^{\alpha}y^{1-\alpha}$, subject to a budget constraint, $B = px + y$. It takes the budget, B, and the relative price of good x in terms of the numeraire good y. Tastes, given by α are taken as given. The derivation starts with the givens and generates a demand curve for the first good. It also determines the demand for the second good, but since the price is fixed, it is not quite right to call that a demand curve. Finally it determines the total utility of the market basket of the quantities of the two goods selected by the consumer.

The basic idea is to solve two equations and two unknowns, x and y and then plug these values into the utility function to determine U. The two equations are first the budget constraint and second the **tangency condition**

$$p = (\frac{\alpha}{1-\alpha})\frac{y}{x}$$

which is derived from setting the slope of the indifference curve (on the right) equal to the slope of the budget constraint (on the left) and canceling both negative signs.

Let's see how this is done:

steps:

1. Multiply through the tangency condition by x to get

$$px = (\frac{\alpha}{1-\alpha})y$$

2. Substitute this px into the budget constraint

$$B = px + y = (\frac{\alpha}{1-\alpha})y + y$$

3. Clear the fraction by multiplying through by $1 - \alpha$

$$(1-\alpha)B = \alpha y + (1-\alpha)y$$

4. Simplify to find

$$(1-\alpha)B = y$$

5. Next find x by substituting the y expression into the budget constraint

$$px + (1-\alpha)B = B$$

6. So that the demand function is

$$x = \frac{\alpha B}{p}$$

7. Note that it is (1) downward sloping (2) non-linear (3) shifts to the right with both B and α the share spent on the first good.

8. Finally plug both x and y into the utility function

$$U = (\frac{\alpha B}{p})^{\alpha}[(1-\alpha)B]^{1-\alpha}$$

Tangency condition explained

This tangency condition can also be expressed in words as setting the **marginal utility per dollar** to be the same for all goods. The marginal utility is

$$\begin{aligned} MU_x &= \frac{\alpha U}{x} \\ MU_y &= \frac{(1-\alpha)U}{y} \end{aligned}$$

Divide the first by p and the second by 1 (the price of the numeraire)

$$\begin{aligned} \frac{MU_x}{p} &= \frac{\alpha U}{px} \\ \frac{MU_y}{1} &= \frac{(1-\alpha)U}{y} \end{aligned}$$

or canceling U on both sides

$$\frac{\alpha}{px} = \frac{(1-\alpha)}{y}$$

This is just the tangency condition rearranged

$$px = (\frac{\alpha}{1-\alpha})y$$

Some examples

1. Solve the following consumer's problem with $\alpha = 0.4$, $p = 1$ and $B = 2$

$$\begin{aligned} x + y &= 2 \\ 1 &= \frac{\alpha}{(1-\alpha)} \frac{y}{x} \end{aligned}$$

   Solution is $[y = 1.0\,2, x = 0.8]$.

2. Let the budget be $B = 10$ and the relative price be $p = 0.5$. With a square root utility function, solve the demand for x and$y$

$$\begin{aligned} 0.5 &= \frac{y}{x} \\ 10 &= 0.5x + y \end{aligned}$$

   Solution is $[x = 10.0, y = 5.0]$.

3. From the previous problem solve for utility. Solution: Plug the solution in for x and y

$$U = \sqrt{xy} = \sqrt{10(5)}$$

   to find $U = 7.0711$.

4. With a budget of $B = 10$ and the relative price be $p = 1$ solve the consumer's problem with a Cobb-Douglas for $\alpha = 0.5$.

$$\begin{aligned} 0.5 &= \frac{y}{x} \\ 10 &= x + y \end{aligned}$$

   Solution is: $[1, x = 5, y = 5.0]$

5. From the previous problem solve for utility. Solution: Plug the solution in for x and y

$$U = \sqrt{xy}$$

   to find $U = 5$.

6. Solve the system with a Cobb-Douglas with $\alpha = 0.3$, $B = 10$ and $p = 0.5$

$$\begin{aligned} 0.5 &= \frac{0.3}{0.7} \frac{y}{x} \\ 10 &= px + y \end{aligned}$$

   Solution is: $[x = 6.0, y = 7.0]$.

7. From the previous problem solve for utility. Solution: Plug the solution in for x and y

$$U = 6^{0.3}7^{0.7}$$

to find $U = 6.6837$.

8. Solve the same system with a Cobb-Douglas with $\alpha = 0.7$ with $p = 0.5$. The solution is

$$\begin{aligned} 0.5 &= \frac{0.7}{0.3}\frac{y}{x} \\ 10 &= px + y \end{aligned}$$

Solution is $[x = 14.0, y = 3.0]$.

9. From the previous problem solve for utility. Solution: Plug the solution in for x and y

$$U = 14^{0.7}3^{0.3}$$

to find $U = 8.8191$.

10. Solve the system with a Cobb-Douglas with $\alpha = 0.7$ and $p = 1$. The solution is

$$\begin{aligned} 1 &= \frac{0.7}{0.3}\frac{y}{x} \\ 10 &= x + y \end{aligned}$$

Solution is $x = 7.0, y = 3.0$

11. From the previous problem solve for utility. Solution: Plug the solution in for y and y

$$U = 7^{0.7}3^{0.3}$$

or $U = 5.43$. The higher price makes the consumer worse off.