Kummer classes and Anabelian geometry

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ABSTRACT. These notes comes from the Super QVNTS: Kummer Classes and Anabelian geometry. Any virtues in the notes are to be credited to the lecturers and not the scribe; however, all errors and inaccuracies should be attributed to the scribe. That being said, I apologize in advance for any errors (typo-graphical or mathematical) that I have introduced. Many thanks to Taylor Dupuy, Artur Jackson, and Jeffrey Lagarias for their wonderful insights and remarks during the talks, Christopher Rasmussen, David Zureick-Brown, and a special thanks to Taylor Dupuy for his immense help with editing these notes. Please direct any comments to `jmorrow4692@gmail.com`.

The following topics were not covered during the workshop:
- mono-theta environments
- conjugacy synchronization
- log-shells (4 flavors)
- combinatorial versions of the Grothendieck conjecture
- Hodge theaters
- kappa-coric functions (the number field analog of étale theta)
- log links
- theta links
- indeterminacies involved in [Moc15a, Corollary 3.12]
- elliptic curves in general position
- explicit log volume computations

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1. On Mochizuki's approach to Diophantine inequalities

Lecturer: Kiran Kedlaya

Disclaimer. The speaker makes no claim to the validity to the statements, and this talk is "for entertainment purposes only".

Let E/F be an elliptic curve over a number field F. For all $\varepsilon > 0$, we would like to prove a Szpiro looking inequality

$$\frac{1}{6}\log|\Delta| \leq (1+\varepsilon)(\log N_E + \log|\mathfrak{d}_F|) + C(\varepsilon),$$

where $\mathfrak{d}_F$ is the different ideal of F and C is some small value depending on ε. In [Moc15c, Theorem 1.10], Mochizuki requires some subtle restrictions on E/F such that E/F is semi-stable and has good reduction above 2, and using [Moc08a], he can bootstrap this restrictive version of the Szpiro inequality to all elliptic curves over F.

Let $\mathfrak{q}$ be the divisor which computes the Tate parameter at primes of bad reduction, and let be ℓ some convenient prime. We can rewrite the above inequality as

$$\frac{1}{6}\log\mathfrak{q} \leq \left(1+\frac{g(F)}{\ell}\right)(\log|N_E| + \log|\mathfrak{d}_F|) + \ell\cdot h(F) \tag{1.0.1}$$

where $g(F)$, $h(F)$ are some functions involving the degree of $F/\mathbf{Q}$ and the latter function is linear in ℓ. We will employ Θ be some set whose volume equals the right hand side of (1.0.1); Kedlaya refers to Θ as a "smeared out version of $\mathfrak{q}$."

The idea is to try to prove this inequality involving degrees of arithmetic divisors by promoting this to an inequality of log-volumes. To find such a statement which gives the comparison between sets and volumes, we refer the reader to [Moc15b, Theorem 3.11], and [Moc15b, Corollary 3.12]. These results state that inside a suitable "measure space", we have

$$\log\mu(\mathfrak{q}) \leq \log\mu(\Theta) \tag{1.0.2}$$

where both volumes correspond to arithmetic data, which yields our desired Diophantine inequality.[1]

1.1. Removing the veil.

We now try and remove the air quotes around measure space. First, we construct the ambient space via taking the product over all primes p

$$\prod_{p\in S_Q}\prod_{j=1}^{(\ell-1)/2}(\text{some } \mathbf{Q}_p\text{-vector space})$$

with the product measures and the multiplication on the $\mathbf{Q}_p$-vector spaces changes the log-volume by $-\log p$. Notice that $(\ell-1)/2$ is the index from Definition 7.5 appears because we really get a bundle of $\mathfrak{q}$ e.g.,

$$\left\{\underline{\underline{q}}^{j^2}\right\}_{j=1,\ldots,(\ell+1)/2}^{(\ell+1)/2}.$$

It would be really helpful to have a global ℓ-Frobenius on E/F since it would allow us to compare arithmetic degrees for E and their Frobenius pullback. One of the goals of

[1] Taylor: The magic inequality actually comes from properties of so-called "log-shells".

inter-universal Teichmüller theory is to give a construction which has the same numerical effect. Thus, one should think about the strategy as we can construct $\mathfrak{q}$ from E in some reasonable way e.g., the action takes place in scheme theory, and by contrast, we construct Θ in an unreasonable way. The unreasonable thing to do is to replace E with a Hodge theater, which remembers a lot of data of and is roughly just a collection of Frobenioids. Formally, we apply the "ℓ^{th}-power Frobenius" and attempt to "reconstruct the resulting elliptic curve".[2]

This reconstruction is in the spirit of anabelian geometry, and although it does not reconstruct E, it produces *some* answer; our goal is to understand what this reconstructed object is (since it is not our original object). In doing so, we encounter mild[3] indeterminacies, meaning that if we try to reconstruct something that actually exists, we won't totally recover the object but rather the object up to some ambiguity i.e., up to ± 1 as above. It is tempting to think that this reconstruction allows one to deduce that $\mathfrak{q} \subseteq \Theta$ inside of these measure spaces.

Warning. The assertion that $\mathfrak{q} \subseteq \Theta$ is never stated in the IUT papers, and Mochizuki states that such an assertion "constitutes a gross misrepresentation" of the content of IUT.

The correct statement is the the region determined by the Θ-pilot object contains a region whose (global) log-volume is equal to the log-volume of the region determined by the $\mathfrak{q}$-pilot object. From this assertion, one can conclude the inequality (1.0.2) and whence (1.0.1). During this conference and within these notes, we hope to exposit some of the underlying ideas and motivations for Mochizuki's work.

2. Why the ABC Conjecture?

Lecturer: Carl Pomerance

The slides for this talk can be found here.

3. Kummer classes, cyclotomes, and reconstructions (I/II)

Lecturer: Kirsten Wickelgren

3.1. **Basic setup.** The fundamental group $\pi_1(X, x)$ classifies covering spaces/maps of a space X. For a scheme X over $\operatorname{Spec} K$, we can consider finite étale maps instead of covering maps. Moreover, we have the étale fundamental group $\pi_1^{\text{ét}}(X, \overline{x})$

$$\pi_1^{\text{ét}}(X, \overline{x}) := \operatorname{Aut}(F_{\overline{x}})$$

where $\overline{x}\colon \operatorname{Spec} \Omega \to X$ for Ω a separably closed field and $F_{\overline{x}}$ is the fiber functor associated to $\overline{x}$.

[2]Taylor: The ℓ^{th} power here is actually the so-called theta link between Hodge theaters and the to "reconstruct the resulting elliptic curve" we mean reconstruct Tate parameters of the elliptic curve. This is the process of relating the so-called theta-pilot objects to so-called q-pilot objects which encode a indirect and direct computation of the Tate parameters respectively.

[3]The word mild is very misleading. A priori, the indeterminacies are far from mild, and in fact, they are horrible. It is a theorem to reduce this to a mild indeterminacies. Taylor: All the indeterminacies, are actually semigroup actions. This is what he means by mild.

For X a locally Noetherian, connected, normal scheme, we can take a geometric point $\eta\colon \operatorname{Spec}\Omega \to X$ whose image is the generic point η, then

$$\pi_1(X,\eta) \cong \mathrm{Gal}(F/K(x)),$$

where $K(x)$ is the function field of X and F is the maximal unramified extension of $K(X)$. The inclusion of fields

$$K(X) \subset K^{\mathrm{sep}}(X) \subset F$$

yields the short exact sequence of Galois groups

$$1 \longrightarrow \mathrm{Gal}(F/K^{\mathrm{sep}}(X)) \longrightarrow \mathrm{Gal}(F/K(X)) \longrightarrow \mathrm{Gal}(K^{\mathrm{sep}}(X)/K(X)) \longrightarrow 1,$$

which one can rewrite as

$$1 \longrightarrow \pi_1(X_{K^{\mathrm{sep}}}) \longrightarrow \pi_1(X,\eta) \longrightarrow G_K \longrightarrow 1.$$

Remark 3.2. For X quasicompact and geometrically connected over K, we have the homotopy exact sequence

$$1 \longrightarrow \pi_1(X_{\overline{K}}) \longrightarrow \pi_1(X) \longrightarrow G_K \longrightarrow 1,$$

and notice the disappearance of the base points.

Our goal is to compare the above situation over K with that over $\mathbf{C}$. For $X/\mathbf{C}$ a compact Riemann surface, the Riemann existence theorem for covering maps asserts that

$$\pi_1^{\text{ét}}(X) \cong \widehat{\pi_1^{\text{top}}(X)},$$

where the right hand side denotes the profinite completion with respect to the finite coverings.

3.3. **Inertia and decomposition groups.** Let U be a geometrically connected, smooth (hyperbolic[4]) curve over a field K of characteristic 0. Let X be a smooth compactification. For x a closed point, let S_x be the set of places of K above x. For $y \in S_x$, a finite étale cover $Y \to U$ and x a closed point in $X \setminus U$, we also have an induced place on the function field Y. Note that the Galois group of K acts on the places of U and will preserve the places of $X \setminus U$. We now give some important definitions.

Definition 3.4. We call the places of K above X which are fixed under the action of the Galois group of K the cuspidal places.

Definition 3.5. The decomposition group D_y for $y \in S_x$ is the stablizier of y in $\pi_1(U)$.

Definition 3.6. The inertia group I_y equals $D_y \cap \pi_1(U_{\overline{K}})$.

Remark 3.7. We have that:

(1) D_y is conjugate to $D_{y'}$ in $\pi_1(U)$ if and only if y and y' lie above the same place in X;
(2) I_y is conjugate to $I_{y'}$ in $\pi_1(U_{\overline{K}})$ if and only if y and y' lie above the same place in $X_{\overline{K}}$.

[4]The toplogical Euler characteristic $\chi(U(\mathbf{C})) < 0$.

3.8. **Reconstruction of decomposition and inertia groups.** Now, we want to reconstruct the cuspidal inertia and decomposition groups from the projection $\pi_1(U) \to G_K$, which follows from work of Nakamura, Tamagawa, and Mochizuki. We first prove the following auxiliary lemma.

Lemma 3.9. *The normalizer* $N_{\pi_1(U_{\overline{K}})}(I_y) = I_y$.

Proof. By the Riemann existence theorem,

$$\pi_1(U_{\overline{K}}) \cong \widehat{\pi_{g,r}}$$

where $\pi_{g,r}$ is the fundamental group for a genus g Riemann surface with r puctures e.g.,

$$\pi_{g,r} := \left\{ \alpha, \beta, \gamma_j : 1 \leq i \leq g, 1 \leq j \leq r \middle| \prod_{i=1}^{g} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \prod_{j=1}^{r} \gamma_j = 1 \right\}.$$

We have that I_y is a closed cyclic subgroup, namely $I_y = \langle h\gamma_j h^{-1} \rangle$ where $h \in \pi_1(U_{\overline{K}})$.

Basic group theory implies that for $r \geq 2$, then $h\gamma_j h^{-1} = \sigma$ can be chosen to be a free generator, and hence the centralizer of σ, $C_{\pi_1(U_{\overline{K}})}(\sigma)$ is simply $\langle \sigma \rangle$. If $h' \in N_{\pi_1(U_{\overline{K}})}(\sigma)$ then

$$h'\sigma h'^{-1} = \sigma^a \quad \text{where } a \in \widehat{\mathbf{Z}},$$

and if we take the abelianization, we have that $a = 1$. Furthermore, we have that

$$N_{\pi_1(U_{\overline{K}})}(I_y) = C_{\pi_1(U_{\overline{K}})}(I_y) = I_y.$$

For $r = 1$, suppose $h \in N_{\pi_1(U_{\overline{K}})}(I_y)$, we can choose a proper, open $H \subset \pi_1(U_{\overline{K}})$ such that $H \supset h$ and $H \supset I_y$ e.g., one can take

$$H \supset [\pi_1(U_{\overline{K}}), \pi_1(U_{\overline{K}})].$$

Since $I_y \subset H$, the covering space corresponding to H is not totally ramified at y, and thus $r \geq 2$ and hence we arrive back at our previous case. □

Proposition 3.10. *Let y be a cuspidal place. Then $D_y = N(I_y)$ is the normalizer of the inertia group in* $\pi_1(U)$.

Proof. For x a closed point of $X_{\overline{K}} \setminus U_{\overline{K}}$, let

$$\mathscr{I}_x = \bigcup_{y \in S_x} I_y.$$

For $x \neq x'$, then $\mathscr{I}_x \cap \mathscr{I}_{x'} = \{1\}$. Since $\pi_1(U_{\overline{K}})$ is normal in $\pi_1(U)$, we have that I_y is normal inside D_y, so $D_y \subset N(I_y)$. Thus, it suffices to show that for the projection map

$$p \colon \pi_1(U) \to G_K$$

, the image of $p(D_y)$ equals $p(N(I_y))$.

Suppose to the contrary that there exists $g \in p(N(I_y)) \setminus p(D_y)$. This implies that g permutes the residue field e.g., it does not fix the point x so $gx = x' \neq x$. This implies that

$$g\mathscr{I}_x g^{-1} = \mathscr{I}_{x'} \cap \mathscr{I}_x = \{1\},$$

which contradicts the fact that there exists a lift of g such that $gI_y g^{-1} \subset I_y$ which follows from $I_y \subset \mathscr{I}_x$ and Lemma 3.9. □

Thus, we have reduced our original problem to reconstructing the inertia groups of cuspidal place, which Nakamura accomplished using Deligne's theory of weights. The following theorem of Nakamura explicitly describes the inertia groups of cuspidal points in terms of group theoretic and Galois properties.

For $J \subset \pi_1(U_{\overline{K}})$ self-normalizing and cyclic, we have a short exact sequence

$$1 \longrightarrow J \longrightarrow N_{\pi_1(U)}(J) \longrightarrow G_K \longrightarrow 1.$$

Theorem 3.11 (Nakamura). *The inertia groups of the points $(X \setminus U)(\overline{K})$ are exactly the nontrivial self-normalizing cyclic subgroups $J \subset \pi_1(U_{\overline{K}})$ such that the map*

$$\begin{aligned} \chi_J \colon G_K &\longrightarrow \operatorname{Aut} J / \operatorname{Inn} J \\ g &\longmapsto gjg^{-1} \end{aligned}$$

is the cyclotomic character, and such that $p(N_{\pi_1(U)} J)$ is open in G_K.

The rough idea of the proof is that we want to consider the difference between α_i, β_i and γ_j in terms of the weight filtration. By applying to étale homotopy $H_\bullet^{\text{ét}}((-)_{\overline{K}}, \mathbf{Z}/\ell\mathbf{Z}) =: H_\bullet(-)$ to $(X_{\overline{K}}, U_{\overline{K}})$, we have the long exact sequence

$$0 = H_2(U) \longrightarrow H_2(X) \cong \mathbf{Z}_\ell(1) \longrightarrow H_2(X,U) \longrightarrow H_1(U) \longrightarrow H_1(X) \cong T_\ell(\operatorname{Jac} X_{\overline{K}}) \longrightarrow H_1(X,U) = 0,$$

where we have that

$$H_2(X,U) \cong \bigoplus_{x \in (X\setminus U)(\overline{K})} H_2(X, X\setminus x) \cong \bigoplus_{x \in (X\setminus U)(\overline{K})} H_2(T_x X, T_x X \setminus \{0\}) \cong \bigoplus_{x \in (X\setminus U)(\overline{K})} \mathbf{Z}_\ell(1).$$

This is the so-called Nakamura sequence, which assists in the proof.

4. Kummer classes, cyclotomes, and reconstructions (II/II)

Lecturer: David Zureick-Brown

Let $\star$ denote a word in the set {relative, absolute, semi-absolute}. In this talk, we will discuss $\star$ anabelian geometry. The overarching theme is to recover an object X from naturally attached gadgets e.g., a curve X from $\pi_1(X)$, a number field from G_K, or a scheme X from a topos $\mathbf{Shv}\, X$. Here is the general yoga for each type of anabelian geometry:

$$\begin{aligned} &\text{(Relative)} && \rightsquigarrow && \pi_1(X) \to G_K, \\ &\text{(Absolute)} && \rightsquigarrow && \pi_1(X), \\ &\text{(Semi-absolute)} && \rightsquigarrow && \pi_1(X_{\overline{K}}) \subset \pi_1(X). \end{aligned}$$

In this lecture, we will discuss some anabelian phenomena occurring in number theory, topology, and category theory.

4.1. Galois groups and number fields.

A starting point for anabelian geometry is the following theorem concerning Galois groups and number fields.

Theorem 4.2 (Neukirch-Uchida). *Let L, K be number fields. Then the following hold:*

1. *If $G_K \cong G_L$ as profinite groups, then $L \cong K$ as fields.*
2. *More precisely, for every profinite group isomorphism $\Phi \colon G_K \to G_L$, there exists a unique field isomorphism $\phi \colon L^{\text{sep}} \to K^{\text{sep}}$ defining Φ i.e., such that*

$$\Phi(g) = \phi^{-1} \circ g \circ \phi \quad \forall\, g \in G_K.$$

In particular, $\phi(L) = K$. *Therefore, the natural map from*

$$\mathrm{Isom}(L, K) \longrightarrow \mathrm{Out}(G_L, G_K)$$

is an isomorphism.

N.B. Note that this is false for local fields via a theorem of Jordan-Ritter.

Much of Mochizuk's work, before IUT, was concerned with anabelian geometry in the 0 and 1 dimensional case. In particular, he proved that Theorem 4.2 is true for local fields by rigidifying, or better keeping track of extra local data coming from the upper ramification filtration,meaning that

$$\mathrm{Isom}_{\mathbf{Q}_p}(L, K) \cong \mathrm{Out}_{\mathrm{Fil}}(G_L, G_K).$$

Mochizuki also proved that for X, Y hyperbolic curves over a number field, then

$$\mathrm{Hom}^{\mathrm{dom.}}(X, Y) \longrightarrow \mathrm{Hom}^{\mathrm{op.,out.}}_{G_K}(\pi_1(X), \pi_1(Y))$$

is an isomorphism.

4.3. **Topological.** Let X, Y be topological spaces. We can associated to each space a category of sheaves $\mathbf{Shv}(X)$ and $\mathbf{Shv}(Y)$. For a morphism $f\colon X \to Y$, we have an adjoint pair of morphisms (f_*, f^{-1}) of $\mathbf{Shv}\,X$ and $\mathbf{Shv}\,Y$; we shall simply call this a geometric homomorphism. There exists a map from

$$\mathrm{Hom}(X, Y) \longrightarrow \mathrm{Mor}(\mathbf{Shv}(X), \mathbf{Shv}(Y))/\sim$$

where $\sim$ is an equivalence relation induced by natural transformations. Note that if X, Y are sober i.e., each irreducible, closed subspace of X, Y has a unique generic point, then the above map is an isomorphism.

Example 4.3.1. Consider the following setup. Let $X = \{\bullet\}$ be a single point and Y be a sober space. Then

$$\mathrm{Hom}_{\mathbf{Top}}(X, Y) = \mathrm{Mor}(\mathbf{Shv}(X), \mathbf{Shv}(Y)) = \mathrm{Mor}(\mathbf{Sets}, \mathbf{Shv}(Y)) = |\widetilde{Y}|$$

via stalks and skyscraper sheaves.

We can put a topology on the set $|\widetilde{Y}|$. Let $U \subset Y$ be open, then

$$|\widetilde{U}| = \left\{ \mathrm{pt} = (f_*, f^{-1}) \in |\widetilde{Y}| : |f^{-1}h_U| = 1 \right\}.$$

This gives an adjoint functor from this topos e.g., an adjoint pair $(|\cdot|, \tilde{\ })\colon$ **Top** $\longrightarrow$ **Local.**, where **Local.** is a topos where the site is an actual topological space, mapping $Y \mapsto |\widetilde{Y}|$. Since Y is sober, then the above map is sober, and hence we can reconstruct our set Y from a topos.

4.4. **Categorical.** We now describe an anabelian phenomena involving derived categories coming from the work of Bondal-Orlov.

Theorem 4.5. *Let X, Y be projective varieties over $\mathbf{C}$ and assume that ω_X, ω_Y are ample or anti-ample e.g., X, Y are Fano or of general type. Suppose that $\phi\colon D^b_{\mathit{Coh}}(X) \cong D^b_{\mathit{Coh}}(Y)$, then $X \cong Y$.*

We now discuss the idea of the proof. Let $E \in D^b_{\mathbf{Coh}}(X)$ such that $E \cong E \otimes \omega_X[-n]$ for some n, $\mathrm{Hom}(E, E) \cong \mathbf{C}$, and $\mathrm{Hom}(E, E[i]) = 0$ for $i < 0$. Then there exists some point $x \in X$ and $i \in \mathbf{Z}$ such that $E \cong \mathscr{O}_X[i]$. There is a similar characterization of Pic. Note that for $f \colon \mathscr{L}_1 \to \mathscr{L}_2$,

$$\operatorname{div} f = \{x \in X : \{\mathrm{Hom}(\mathscr{L}_1, \mathscr{O}_x) \to \mathrm{Hom}(\mathscr{L}_2, \mathscr{O}_x)\} = 0\},$$

which recovers set theoretic and topological information, and if one works a bit harder, then one can recover the scheme structure. We refer the reader to [BO02] for more details.

4.6. **Arithmetic geometry.** We now discuss an example from Tamagawa. Let k be finite, and X/k an affine, smooth, geometrically connected curve, let Σ denote the set of cusps, and $K = \kappa(X)$. We loosely describe how one can reconstruct K from $G_K \to G_k$.

First, one can detect whether $X(k) \neq \emptyset$ purely group theoretically. Indeed, if X is proper, then using the Lefschetz trace formula via $\pi_1(X)^{\mathrm{ab},\ell}$ and Frob lies in this group. Second, one can characterize the decomposition group of points inside G_K. Third, we can describe the multiplicative group via the map

$$1 \longrightarrow K^\times \longrightarrow \left(\prod_{x \in X} \widehat{K_x}/\widehat{\mathscr{O}_x}^\times \times \prod_{x \in \Sigma} \widehat{K_x}^\times\right) \longrightarrow \pi_1(X)^{\mathrm{ab}} \longrightarrow 1.$$

The more difficult procedure is to reconstruct K from $K^\times$.

Remark 4.7. If we have $x \colon X \to \operatorname{Spec} k$, which gives a map $\pi_1(X) \to G_K$ and a section, functorially, yields a section from $\alpha \colon G_K \to \pi_1(X)$. Given some section α, α arises from a point if and only if for every open $H \subset \pi_1(X)$ containing the image of α, $Y^H(k) \neq \emptyset$ where Y^H is the finite étale covering corresponding to H.

4.8. **Why the IUT formalism?** We actually want to prove a hybrid Szpiro/Vojta conjecture, meaning that we are comparing the discriminant and the conductor, which we can reduce to a local situation. We can reduce to the case where $E/\mathbf{Q}_p$ has split semistable reduction, and hence we are led to the Tate curve. Then we can compute $\nu_p(q)$, where q is the Tate parameter in multiple ways. To do so, we actually work on covers of the Tate curve which are better suited for anabelian methods; this will become clearer in a few lectures. It appears that a basic strategy is to compute that valuation of the Tate parameter $\nu(q_\nu)$ in two different ways and to study their disparity.

5. Overflow session: Kummer classes

Lecturer: Taylor Dupuy

In this overflow session, we want to discuss Kummer classes, which will be relevant for Section 7. Roughly speaking, we want purely algebraic methods to extract information concerning the valuation of the Tate parameter q associated to an elliptic curve over a p-adic field. Let X be an affine curve or a non-Archimedean space over K. Let $\pi_X := \pi_1(X, x)$ denote the profinite or tempered fundamental group; if we want to consider covers of non-Archimedean spaces, we need the tempered fundamental group to ensure that the covering spaces are isomorphic to $\mathbf{Z}$ and not to $\widehat{\mathbf{Z}}$.

We now discuss the content coming from T. Dupuy's vlog post to give a more detailed discussion about Kummer classes, in particular Kummer classes of functions. Let K be a field and X/ an affine scheme, so we have the notion of functions. The construction we seek is:

$$\kappa\colon \mathscr{O}^{\times}(X) \longrightarrow H^1(X, \widehat{\mathbf{Z}}(1)_X)$$
$$f \longmapsto \kappa_{X,f},$$

where $\widehat{\mathbf{Z}}(1)_X$ is a cyclotome (c.f. Definition 5.1) and $\kappa_{X,f}$ will be called the Kummer class of f. The purpose of this construction is to find a group theoretic method to evaluate functions.

Recall the baby Kummer sequence, with the following sequence of group schemes over K

$$1 \longrightarrow \mu_n \longrightarrow \mathbf{G}_m \longrightarrow \mathbf{G}_m \longrightarrow 1.$$

We want to view the sequence of group schemes as étale sheaves of groups $\mathbf{Shv}(X_{\text{ét}}, \mathbf{Grp})$. To accomplish this, we base change to X yielding

$$1 \longrightarrow \mu_{n,X} \longrightarrow \mathbf{G}_{m,X} \xrightarrow{[n]} \mathbf{G}_{m,X} \longrightarrow 1.$$

N.B. We would like to say when the above sequence is exact after taking sections. If one picks X and takes an X-section of $\mathbf{G}_{m,X}$, then there is an étale cover $X' \to X$ such that the restriction of that section to $\mathbf{G}_{m,X}(X')$ is in the image. With such a choice, we have the natural action of $\pi_1^{\text{ét}}(X)$ on X' via deck transformations and whence we have the exact sequence

$$1 \longrightarrow \mu_{n,X}(X') \longrightarrow \mathbf{G}_{m,X}(X') \xrightarrow{[n]} \mathbf{G}_{m,X}(X') \longrightarrow 1.$$

of $\pi_1^{\text{ét}}(X)$-module. Note that taking section is generally not exact, but it becomes exact after a particular choice; one should think of this as a "call and response" kind of thing where each section requires its own X'. Moreover, we can consider the two sequences with X' the universal cover as living in the category of group schemes and the category of $\pi_1^{\text{ét}}(X)$-modules. In fact, for finite group schemes and limits of finite group schemes the group cohomology with the étale fundamental group will coincide with the étale cohomology.

We consider the long exact sequence associated to the above; we also omit the subscript of X. Consider the connecting morphism in the long exact sequence, which we denote by

$$\kappa_X^{(n)}\colon H^0(X, \mathbf{G}_m) \longrightarrow H^1(X, \mu_n).$$

Notice that $H^0(X, \mathbf{G}_m) = \mathscr{O}_X^{\times}$ and so we back to the baby Kummer sequence.

Definition 5.1. We define a cyclotome of X/k to be the inverse limit of group schemes

$$\widehat{\mathbf{Z}}(1)_X := \varprojlim_n \mu_{n,X}.$$

A true fact is that

$$\varprojlim H^1(X, \mu_{n,x}) = H^1(X, \varprojlim \mu_{n,X}) = H^1(X, \widehat{\mathbf{Z}}(1)_X),$$

so we could alternatively define a cyclotome as the group scheme witnessing the above limit. Thus, we get a Kummer map

$$\widehat{\kappa}_X\colon H^0(X, \mathbf{G}_m) \longrightarrow H^1(X, \widehat{\mathbf{Z}}(1)_X) \tag{5.1.1}$$

by taking limits, so

$$\widehat{\kappa}_X := \varprojlim_n \kappa_x^{(n)}.$$

To summarize, we have the map

$$\begin{aligned} \mathscr{O}^\times(X) &\longrightarrow H^1(X, \widehat{\mathbf{Z}}(1)) \\ f &\longmapsto \widehat{\kappa}_X(f) = \kappa_f, \end{aligned}$$

where we have that $H1(X, \widehat{\mathbf{Z}}(1)_X) \cong H^1(\pi_1^{\text{ét}}(X), \widehat{\mathbf{Z}}(1)_X)$.

Under the hood, we have described a natural transformation

$$\widehat{\kappa}\colon H^0(-, \mathbf{G}_m) \longrightarrow H^1(-, \widehat{\mathbf{Z}}(1_X))$$

for functors $\mathbf{Sch}_K \to \mathbf{Ab}$. If $\varphi\colon Y \to X$, we get a diagram

$$\begin{array}{ccc} H^0(X, \mathbf{G}_m) & \xrightarrow{\widehat{\kappa}_X} & H^1(X, \widehat{\mathbf{Z}}(1)_X) \\ \varphi^* \downarrow & & \downarrow \varphi^* \\ H^0(Y, \mathbf{G}_m) & \xrightarrow{\widehat{\kappa}_Y} & H^1(Y, \widehat{\mathbf{Z}}(1)_Y) \end{array}$$

and hence a compatibility among Kummer maps. This functoriality property allows us to evaluate functions because it will illustrate that evaluation of functions of points factors through cohomology, and then our goal will be to reconstruct the cohomology classes along with the evaluations.

We now want to discuss the group theoretic evaluation of functions. As an example from the above, let $Y = \operatorname{Spec} K$ and consider rational points

$$a\colon \operatorname{Spec} K \longrightarrow X.$$

Plugging this into our above machinery, we have

$$\begin{array}{ccc} \mathscr{O}^\times(X) & \xrightarrow{\widehat{\kappa}_X} & H^1(X, \widehat{\mathbf{Z}}(1)_X) \\ a^* \downarrow & & \downarrow a^* \\ K^\times & \xrightarrow{\widehat{\kappa}_K} & H^1(K, \widehat{\mathbf{Z}}(1)_K) \cong \widehat{K}^\times. \end{array}$$

The left-vertical map take $f \to f(a)$ e.g., just the usual evaluation of functions, and so the commutativity of diagram states that

$$\widehat{\kappa}_X(f(a)) = a^*(\widehat{\kappa}_{X,f}) \in H^1(K, \widehat{\mathbf{Z}}(1)).$$

This is the relationship between evaluation of points and Kummer classes.

Recall that we have the isomorphisms

$$\delta_n := \kappa_K^{(n)}\colon K^\times / K^{\times^n} \cong H^1(K, \mu_)$$

which induced the map

$$\widehat{\delta}\colon \widehat{K}^\times \longrightarrow H^1(K, \widehat{\mathbf{Z}}(1)).$$

Putting everything together, we have the Kummer theoretic evaluation e.g.,

$$f(a) = \widehat{\delta}^{-1}(a^* \widehat{\kappa_{X,f}}) \in \widehat{K}^\times.$$

In particular, we have that $f(a) \in K^\times \hookrightarrow \widehat{K}^\times$.

We conclude with another important component to Kummer classes. For

$$\kappa_f \in H^1(\pi_1^{\text{ét}}(X), \widehat{\mathbf{Z}}(1)),$$

the map

$$\pi_1^{\text{ét}}(X) \longrightarrow G_K$$

and a section $s\colon G_K \to \pi_1^{\text{ét}}(X)$ induced by a K-rational point of the curve induces a map

$$H^1(\pi_1^{\text{ét}}(X), \widehat{\mathbf{Z}}(1)) \longrightarrow H^1(G_K, \widehat{\mathbf{Z}}(1)) \cong \widehat{K}^\times$$
$$\kappa_f \longmapsto s^*\kappa_f.$$

Notice that the section is only well-defined up to conjugacy, and so we have some conjugacy indeterminacy in the above map, which we need to handle. The eventual goal is to just use the image of s (if s is an actual morphism of groups not an outer automorphism)

$$\mathrm{Im}(s) \cong D_{\widetilde{X}/X} \subset \pi_1^{\text{ét}}(X)$$

to evaluate the κ_f where $D_{\widetilde{X}/X}$ is the decomposition group of a non-cuspidal point. The idea is that given the functorial construction then a genuine section gives us this map, but we have the conjugacy indeterminacy, which we need to deal with. We want to apply this method, and eventually we will just use the decomposition groups, which can be given an anabelian interpretation.

In Section 7, we will consider the Kummer class of the Jacobi theta function.

6. Introduction to model Frobenioids

Lecturer: Andrew Obus

By way of introduction, Mochizuki loosely defines a Frobenioid as a category theoretic abstraction of divisors or line bundles on a geometry object. Our main example will be an abstract category which encodes étale coverings and information concerning divisors.

6.1. **Monoids.** The inspiration for this perspective comes from logarithmic geometry. Let M be a commutative monoid with 0. For any M, we have the groupification M^{gp}, which is universal with respect to group-like monoids e.g., groups. For a simple example, think $\mathbf{N}$ as a additive monoid and $\mathbf{Z}$ as the groupification.

Definition 6.2. We call M divisorial if it satisfies[5]

(1) 0 is the only invertible element (sharp),
(2) $M \hookrightarrow M^{gp}$ (integral),
(3) If $a \in M^{gp}$, $n \in \mathbf{N}$, $na \in M$, then $a \in M$ (saturated).

6.3. **Three types of Frobenioids.** We will discuss three types of Frobenioids, which will increase in complexity.

6.3.1. *A basic Frobenioid.* Let M be a commutative monoid, $\mathbf{N}$ a multiplicative monoid, then $M \rtimes \mathbf{N}$ is a noncommutative basic Frobenioid, $(\mathbf{F}_M)$ defined by

$$(b, m) \cdot (a, n) = (b + ma, mn).$$

[5] One should think of this as the effective divisor on a variety.

6.3.2. *An elementary Frobenioid.* Notice that a basic Frobenioid was just the semi direct product of two monoids another one. We now want to take a sheafification of a larger category; if we take a category with one element, then we recover the basic Frobenioid by identifying the category with a monoid. Let $\mathcal{D}$ be a category.

Definition 6.4. Consider the contravariant functor from $\mathcal{D}$ to the category of commutative monoids

$$\Phi\colon \mathcal{D} \longrightarrow \mathbf{CMon}$$

sending $(f\colon A \to B) \in \mathrm{Mor}(\mathcal{D})$ to the pull back along f, $(f^*\colon \Phi_B \to \Phi_A) \in \mathrm{Mor}(\mathbf{CMon})$. We call such a Φ a (sheaf) of monoids on $\mathcal{D}$ if it satisfies some mild conditions.

Definition 6.5. We define an elementary Frobenioid as a category $\mathbf{F}_\Phi$ for Φ a sheaf of monoids on $\mathcal{D}$ where the objects are $\mathrm{Obj}(\mathcal{D})$ and the morphisms are defined by

$$\mathrm{Hom}_{\mathbf{F}_\Phi}(A, B) = \{(f, s, n) : f \in \mathrm{Hom}_{\mathcal{D}}(A, B), s \in \Phi_A, n \in \mathbf{N}\},$$

where composition

$$A \overset{(f,s,n)}{\longrightarrow} B \overset{(g,t,m)}{\longrightarrow} C$$

is defined by $(g \circ f, f^*t + ms, mn)$.

Example 6.5.1. Let X be a proper normal variety, S a set of $\mathbf{Q}$-Cartier prime divisors on X, $\mathcal{D}$ is the category of finite separable covers of X. If $A \in \mathrm{Obj}(\mathcal{D})$, then

$$\Phi_A = \{\delta \in \mathrm{Eff}(A) : \delta \text{ } \mathbf{Q}\text{-Cartier, image of } d \text{ under } A \to X \text{ is contained in a sum of divisors of } S\}.$$

If $f\colon A \to B$, then we have a pullback $f^*\colon \Phi_B \to \Phi_A$. We can construct the elementary Frobenioid $\mathbf{F}_\Phi$, as we did above.

N.B.

(1) For $\mathcal{D}$ is category with one object and one morphism, we end up the the same rule for the composition of a basic Frobenioid.
(2) We can create an elementary Frobenioid associated to a number field K where we replace S with finite separable extension and S is the set of Arakelov divisors.

6.5.2. *A pre-Frobenioid.* Let $\mathbf{F}_\Phi$ be an elementary Frobenioid, Φ be divisorial e.g., it maps to divisorial monoids, the base category $\mathcal{D}$ to be totally epimorphic[6] and connected[7], and $\mathcal{C}$ a connected totally epimorphic category.

Definition 6.6. A pre-Frobenioid is a covariant functor from $\mathcal{C} \to \mathbf{F}_\Phi$ with the above assumptions.

Remark 6.7. All of the relevant information of the pre-Frobenioid is contained in the category $\mathcal{C}$, and we can reconstruct this functor to $\mathbf{F}_\phi$ via a reconstruction theorem. Thus, the datum of a pre-Frobenioid is encoded in the category $\mathcal{C}$.

6.7.1. *A Frobenioid.*

Definition 6.8 (Bad definition)**.** A Frobenioid is a pre-Frobenioid satisfying a laundry list of conditions, which we do not discuss; for details, see [Moc08b, Definition 1.3].

[6] All of the morphisms are surjective.
[7] The graph of the morphisms is connected.

6.9. **Main example: Model Frobenioids.** Let $\mathcal{D}$ be a totally epimorphic connected category, Φ a divisorial monoid on $\mathcal{D}$, $\mathbf{F}_\Phi$ the associated elementary Frobenioid, and $\mathbf{B}$ a group-like monoid on $\mathcal{D}$. We have a map

$$\mathrm{Div}\colon \mathbf{B} \longrightarrow \Phi^{\mathrm{gp}}$$

a morphism of monoids on $\mathcal{D}$. We can construct a model Frobenioid $\mathcal{C} \to \mathbf{F}_\Phi$ as follows:

- the objects are (A, α) for $A \in \mathrm{Obj}(\mathcal{D})$ and $\alpha \in \Phi_A^{\mathrm{gp}}$,
- the morphisms $\mathrm{Hom}((A,\alpha),(B,\beta))$ are given by quadruples

  $$(f, s, n, u),$$

  where (f, s, n) is a morphism in $\mathbf{F}_\Phi$ and $u \in \mathbf{B}_A$ such that

  $$n\alpha + s = f^*\beta + \mathrm{Div}(u).$$

  The composition is similar to as we have defined it for $\mathbf{F}_\Phi$, and we have a semi-direct product like operation for u.

Example 6.9.1. In our previous Example 6.5.1, we can take $\mathbf{B}_A = K(A)^\times$.

Example 6.9.2. Suppose we have a morphism $(A, \alpha) \to (A, \beta)$. Then:

- $(\mathrm{id}_A, S, 1, 1)$ forces $\alpha + s = \beta$,
- $(\mathrm{id}_A, 0, n, 1)$ induces $n\alpha = \beta$, and
- $(\mathrm{id}_A, 0, 1, u)$ forces $\alpha = \beta + \mathrm{Div}(u)$.

Remark 6.10. We wish to make a few remarks:

(1) Model Frobenioids are Frobenioids;
(2) We can do the above procedures for number fields;
(3) There exists a tempered analog for all of the above definitions.

We now conclude with a theorem illustrating an anabelian flavor of Frobenioids.

Theorem 6.11 ([Moc08b])**.** *Under certain assumptions, $\mathcal{C} \to \mathbf{F}_\Phi$ can be reconstructed from $\mathcal{C}$.*

7. Theta functions and evaluations

LECTURER: EMMANUEL LEPAGE

Let ℓ be an odd prime number, K_v a p-adic field where $p \neq 2, \ell$, E/K_v and elliptic curve with split multiplicative reduction, and q its Tate parameter. Assume that $\{\mu_{4\ell}, q^{1/2\ell}\} \subset K_v$ e.g., we have some torsion on our base field. Then consider the map $\mathbf{G}_m^{\mathrm{an}} \to E$ and let $X = E \setminus \{P\}$ where P is a point, so X is affine.

Consider the diagrams (with the unorthodox notation):

$$\begin{array}{ccc}
\ddot{Y} = Y[u^{1/2}] & & \mathbf{G}_m^{\mathrm{an}} \\
\downarrow & & \downarrow \\
Y & \hookrightarrow & \mathbf{G}_m^{\mathrm{an}} = \mathrm{Spec}\, K_v[u^{\pm}] \\
\downarrow & & \downarrow \\
X & \hookrightarrow & E
\end{array}$$

Consider the Jacobi theta function $\ddot{\Theta} \in \mathscr{O}^{\times}(\ddot{Y})$ which is define as

$$\ddot{\Theta} := C \sum_{n \in \mathbf{Z}} (-1)^n q^{1/2(n^2+n)} \ddot{u}^{2n+1} = C' \ddot{u} \prod_{n<0} \frac{\ddot{u}^2 - q^n}{q^n} \prod_{n>0} \frac{\ddot{u}^2 - q^n}{\ddot{u}^2}.$$

We also have the functional equations

$$\begin{aligned} &\ddot{\Theta}(u^{-1}) = -\ddot{\Theta}(\ddot{u}) && \ddot{\Theta}(-\ddot{u}) = -\ddot{\Theta}(\ddot{u}) \\ &\ddot{\Theta}(q^{a/2}\ddot{u}) = (-1)^a q^{-a^2/2} \ddot{u}^{-2a} \ddot{\Theta}(\ddot{u}) && \ddot{\Theta}(q^{a/2} i) = q^{-a^2/2} \ddot{\Theta}(i), \end{aligned}$$

where we set $q^{a/2} i$ as the evaluation point and the normalized theta function $\ddot{\Theta}(i) = \pm 1$.

We want an anabelian construction of $\ddot{\Theta}$, which we accomplish by considering its Kummer class. We have a map

$$\mathscr{O}^{\times}(\ddot{Y}) \longrightarrow H^1_{\text{ét,an}}(\ddot{Y}, \widehat{\mathbf{Z}}(1)) \leftarrow H^1(\pi^{\text{temp}}_{\ddot{Y}}, \widehat{\mathbf{Z}}(1)),$$

where π^{temp}_X is the tempered fundamental group, which is a pro-discrete group, and it admits an open subgroup of $\pi^{\text{temp}}_{\ddot{Y}}$.

N.B. The tempered fundamental group allows us to realize, or better see, $\mathbf{Z}$ as a subgroup of $\widehat{\mathbf{Z}}(1)$, which a priori we cannot.

7.1. **Anabelian construction of theta function.** We can consider the Kummer class of the theta function

$$\ddot{\Theta} \longmapsto \kappa\ddot{\Theta},$$

and in fact

$$\kappa\ddot{\Theta} \in H^1(\pi^{\text{temp}}_{\ddot{Y}}, \widehat{\mathbf{Z}}(1)).$$

We want something isomorphic to $\widehat{\mathbf{Z}}(1)$, which can be defined purely in terms of our fundamental group e.g., we need to make a choice of anabelian cyclotome.

Let

$$\Delta^{\text{temp}}_X = \pi^{\text{temp}}_1(X_{\overline{K_v}}),$$

which fits into the short exact sequence

$$1 \longrightarrow \Delta^{\text{temp}}_X \longrightarrow \pi^{\text{temp}}_X \longrightarrow G_{K_v} \longrightarrow 1.$$

We define the following unipotent quotients

$$\begin{aligned} \Delta^{\Theta}_X = \Delta^{[2]}_X = \Delta^{\text{temp}}_X / [\Delta^{\text{temp}}_X, [\Delta^{\text{temp}}_X, \Delta^{\text{temp}}_X]] \\ \pi^{\Theta}_X = \pi^{[2]}_X = \pi^{\text{temp}}_X / [\Delta^{\text{temp}}_X, [\Delta^{\text{temp}}_X, \Delta^{\text{temp}}_X]], \end{aligned}$$

and we have the following diagram

$$\begin{array}{ccc} \pi^{\text{temp}}_X & \twoheadrightarrow & \pi^{\Theta}_X \\ \uparrow & & \uparrow \\ \pi^{\text{temp}}_{\ddot{Y}} & \twoheadrightarrow & \pi^{\Theta}_{\ddot{Y}}. \end{array}$$

The profinite completion of Δ_X^{Θ} can be identified as

$$\widehat{\Delta_X}^{\Theta} = \left\{ \begin{pmatrix} 1 & \widehat{\mathbf{Z}}(1) & \widehat{\mathbf{Z}}(1) \\ 0 & 1 & \widehat{\mathbf{Z}} \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

with subgroups

$$\widehat{\Delta_X}^{\Theta} \supset \Delta^{\Theta} = \left\{ \begin{pmatrix} 1 & \widehat{\mathbf{Z}}(1) & \widehat{\mathbf{Z}}(1) \\ 0 & 1 & \mathbf{Z} \\ 0 & 0 & 1 \end{pmatrix} \right\} \supset \Delta_{\ddot{Y}}^{\Theta} = \left\{ \begin{pmatrix} 1 & 2\widehat{\mathbf{Z}}(1) & 2\widehat{\mathbf{Z}}(1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Definition 7.2. Let

$$\Delta_{\Theta} := \mathrm{Im}([\Delta_X, \Delta_X]) \cong \widehat{\mathbf{Z}}(1).$$

We define Δ_{Θ} to be the interior cyclotome, and note that

$$\Delta_{\Theta} \subset \Delta_{\ddot{Y}}^{\Theta} \subset \pi_{\ddot{Y}}^{\Theta}.$$

If we let $F_2 := \Delta_{\Theta}$, $F_1 := \Delta_{\ddot{Y}}^{\Theta}$, and $F_0 := \pi_{\ddot{Y}}^{\Theta}$, we can use the Leray spectral sequence to give the filtration $F_0 \subset F_1 \subset F_0 = H^1(\pi_{\ddot{Y}}^{\Theta}, \Delta_{\Theta})$. Moreover, we have the following quotients

$$\begin{aligned} F_0/F_1 &:= \mathrm{Hom}(\Delta_{\Theta}, \Delta_{\Theta}) \\ F_1/F_2 &:= \mathrm{Hom}(\Delta_{\ddot{Y}}^{\Theta}/\Delta_{\Theta}, \Delta_{\Theta}) \\ F_2 &= H^1(G_{K_v}, \Delta_{\Theta}) \cong \widehat{K_v^{\times}}. \end{aligned}$$

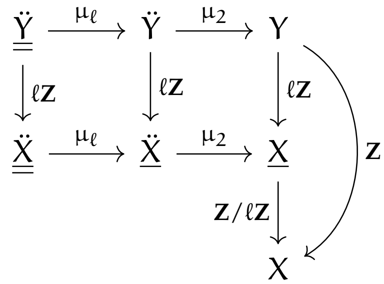
We have an involution on $\ddot{Y}$ mapping $\ddot{u}$ to $\ddot{u}^{-1}$, which is compatible with the filtration and induces the identity on F_0/F_1, $-\mathrm{id}$ on F_1/F_2, and id on F_2. Furthermore, this gives a unique class $\Theta_{\text{ét}}$ up to $\widehat{K_v^{\times}}$ with image of id in F_0/F_1 invariant by involution. Evaluation $\Theta_{\text{ét}} \in F_0$ at $\pm i$ gives us two Galois sections s up to conjugacy from $G_{K_v} \to \pi_{\ddot{Y}}^{\text{temp}}$, and by pulling back $s^*(\Theta_{\text{ét}}) \in \widehat{K_v^{\times}}$ and normalizing $\Theta_{\text{ét}}$ such that $s_i^*(\Theta_{\text{ét}}) \in \{\pm i\}$. These Galois sections can be recovered in an anabelian way, and thus we can realize

$$\Theta_{\text{ét}} \in H^1(\pi_{\ddot{Y}}^{\text{temp}}, \Delta_{\Theta})$$

up to ± 1. The point of this is that we can construct a theta function simply from the fundamental group.

Remark 7.3. Geometrically and canonically, we have that $\Delta_{\Theta} \cong \widehat{\mathbf{Z}}(1)$. Also, if we take the Kummer class of $\Theta_{\text{ét}} \in H^1(\pi_{\ddot{Y}}^{\text{temp}}, \Delta_{\Theta})$, we see that $\Theta_{\text{ét}} = \kappa_{\ddot{\Theta}} \in H^1(\ddot{Y}, \widehat{\mathbf{Z}}(1))$. Note that both of these depend on choice of involution on $\ddot{Y}$.

7.4. **A bit more on the IUT formalism.** In IUT, we do not directly consider $\ddot{\Theta}$, but rather as a p^{th} root. We also have $\ddot{Y}(\ddot{\Theta}^{1/p}) = \underline{\underline{Y}}$ which fits into the zoo of covers:



We set
$$\underline{\underline{\Theta}} = \ddot{\Theta}^{1/p} \in \mathscr{O}^{\times}(\underline{\underline{\ddot{Y}}}),$$
which is defined up to $\mu_{2\ell}$. For $j \in \mathbf{Z}$, we can consider the value of the étale theta function which maps to $\pm q^{j/2} i \in \ddot{Y}(K_v)$ and then by taking pull backs, we have $\mu_{-,j} \in \underline{\underline{\ddot{Y}}}(K_v)$, which is defined up to $\mu_{2\ell}$. Furthermore,
$$\underline{\underline{\Theta}}(\mu_{-,j}) = q^{j^2/2\ell} = \underline{\underline{q}}^{j^2}$$
where $\underline{\underline{q}}^{2\ell} = q$.

Now consider the dual graphs of $\underline{\underline{\ddot{X}}}$ and $\underline{\underline{\ddot{Y}}}$.

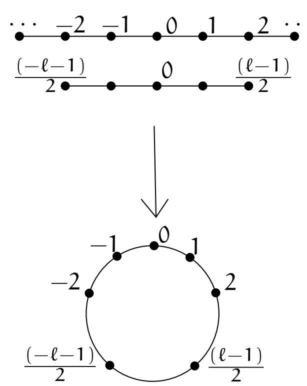


FIGURE 1. Morphism of dual graphs $\Gamma(\underline{\underline{\ddot{Y}}}) \to \Gamma(\underline{\underline{\ddot{X}}})$

N.B. We can also recover the dual graph structures from the tempered fundamental group.

We take
$$\xi = (\mu^{*}_{-,j}(\underline{\underline{\Theta_{\text{ét}}}})_{j=1,\ldots(\ell-1)/2}) \in \prod_{j=1} H^1(G_{K_v}, \Delta_{\Theta}),$$
where we pick our cyclotome in a rigid way, as Mochizuki states.

Definition 7.5. We can now define the Gauss monoid
$$\psi_G = \mathscr{O}^{\times}_{K_v} \xi^{\mathbf{N}} \cong \mathscr{O}^{\times}_{K_v}(\underline{\underline{q}}, \underline{\underline{q}}^{2^2}, \ldots, \underline{\underline{q}}^{((\ell-1)/2)^2}).$$

8. Roadmap of proof

Notes from an email from Taylor Dupuy

Mochizuki's general strategy is to compute the lefthand side of the Szpiro inequality

$$|\Delta| \leq C(\varepsilon) \cdot N_E^{6+\varepsilon},$$

where Δ and N_E are, respectively, the discriminant and the conductor of an elliptic curve E, in two different ways: one directly and one using anabelian geometry. In a bit more detail, the anabelian method reconstructs collections of the Tate parameters up to indeterminacies arising in the form of theta pilots and adelic regions; the word adelic here refers to the cobbling together of local bits of information. We stress that this is not just a local reconstruction, but rather a simultaneous reconstruction of *both* local and global data, in a compatible way. The indirect computation yields the volume appearing on the right hand side of Mochizuki's inequality [Moc15b, Corollary 3.12]. A further explicit adelic volume computation then relates this volume back to the left hand side of the Szpiro inequality imposing a relation on the minimal discriminant.

It is in this explicit adelic volume computation that the log conductor and log different term appear, which has to do with general properies of so called log-shells, which are the measure spaces we are going to compute these log volumes in. The resulting inequality is [Moc15c, Theorem 1.10] (this is the inequality which combines the log volume inequality with the log volume computation). This inequality can then be shown to imply the Vojta conjecture for curves (cf. the end of [Moc15c]); note that this part of the proof utilizes [Moc08a].

We now to describe the "anabelian log volume computation" of the previous paragraph a bit further. First, recall that the valuation of the Tate parameters at non-Archimedean places are the same as the valuation of the discriminant of the elliptic curve. This means that the left hand side of the Szpiro conjecture is encoded in Tate parameters of the elliptic curve at places of bad reduction. For this reason Mochizuki focuses on recovering the valuation of these Tate parameters.

How does one recover the Tate parameters of elliptic curves at bad reduction? Loosely speaking, one does this by looking at special values of special functions at special points. The special function is the Jacobi theta function and the special points are so called evaluation points. These are both reconstructed using anabelian geometry. Here is the idea: functions have associated Kummer classes in $H^1(\pi_X, \widehat{\mathbf{Z}}(1))$, where the Galois module $\widehat{\mathbf{Z}}(1)$ is the Tate module of $\mathbf{G}_m$ and is what is called in Mochizuki's literature a cyclotome (cf. Definition 5.1). Note that many different isomorphic copies of this module appear in many different ways. Mochizuki calls isomorphisms between these copies cyclotomic synchronizations (some copies have better anabelian properties than others).

It turns out that the restriction of these Kummer classes to decomposition groups is the same as evaluation of a function (after using a suitable isomorphism of $H^1(G_K, \widehat{\mathbf{Z}}(1))$ with the completion of the units of K [here K is a finite extension of $\mathbf{Q}_p$]). In our application, the Kummer class of the Jacobi theta function (which Mochizuki calls the étale theta function) and the evaluation points can be given an anabelian interpretation. One of the issues is the ℓ^{th} root of a normalized Jacobi theta function can only be reconstructed up to an orbit (by $\mathbf{Z}/\ell\mathbf{Z} \times \mu_2$) so in order to recover the evaluation more precisely one needs to pay a price. This price is having to evaluate at many points which gives this Gaussian distribution.

What ends up happening is that we actually recover $\ell/2$ evaluation values at a time corresponding to a certain fundamental domain of a group action by $\mathbf{Z}/\ell\mathbf{Z}$. Note that all this Gaussian business is just a result of the functional equation for the Jacobi theta function. Essentially, it means that one recovers not the Tate parameters but powers of the Tate parameters; actually, we are really recovering $\underline{\underline{q}} = q^{1/2\ell}$ up to an $2\ell^{\text{th}}$ root of unity. The zoo of covers we are dealing with are essentially trying to make all these constructions make sense in rigid geometry where square roots are not allowed.

Since we have a way of locally recontructing special values, we want to start gluing these things together using the product formula. After gluing we obtain the so called Gaussian Frobenioids, which appear at the end of [Moc15a]. In [Moc15b], Mochizuki seeks a "multi-radial representation" of these theta values (which have now been encoded as objects of certain "realified global Frobenioids" which are essentially some categorical method for encoding Arakelov divisors). Essentially, one has versions of all these objects reconstructed from various objects (key words are "mono-theta environments", "Frobenioids"). A general theme the following: if you want to reconstruct something from a weaker object (e.g., Galois group of a base field rather than fundamental group of a curve rather than a Frobenioid) you need to introduce indeterminacies (which mean that you only get to reconstruct your objects up to a monoid action or group action). The goal of [Moc15b] is to reconstruct the theta pilot object (these"Gaussian values glued together") in a way that will allow us to compare it with a Tate parameter (categorically encoded as a q-pilot object which is also an object of a certain categorically reconstructed realified arithmetic Frobenioid).

A theta pilot object is just an object of a constant Frobenioid which encodes the generators of a certain (global) Gaussian monoid coming from evaluation (one needs to fiddle a bit here to get the correct behavior with respect to the new indeterminacies introduced, in particular a "log-link" is introduced in the process). What one does then is that one "quotients" two abstract copies of big Hodge theaters encoding q parameters and theta values (this is the theta link) and pushes the theta values to a really weak object which can be shared between two different Hodge theaters. These are the so called monoanalytic etale-like log shells. They are monoanalytic because they use only local Galois groups and they are etale-like because they don't required the full structure of the Frobenioid. This whole process is described in [Moc15b, Theorem 3.11.i]. Here is what is claimed in the finishing touches: one passes to a "holomorphic hull" a subset which contains the orbit of all such regions under the indeterminacies and claims the log-volume of this region gives an upper bound on the log-volume of the region determined by the Tate parameters. In the end it is upper bounds on volumes of this region which gives the right hand side of Mochizuki's [Moc15c, Theorem 1.10] (which implies Vojta). These details are a bit foggy, but we hope these comments shed some light on what is going in in Mochizuki's proof.

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