Linear Programming Optimization, Part 1

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We are now going to switch to a somewhat different problem. You have a handout which is called 'Saving the Endangered Species of Erewhon.' If you haven't read it, you need to read it now. And if you have read it, hold it in front of you while we talk about it a bit because I would like to clarify some of the points in it.

PAUSE the video: Read 'Saving the Endangered Species of Erewhon'

The first point I'd like to clarify is that there are a total of 590 endangered species in this problem. In other words, the 350 species in the valley and the 240 in the highland are completely different species.

What is your objective here? Your objective is to save as many of those plants as possible.

The second point is that there's been a budget crisis in Erewhon, and there is no hope that you are going to get more than your 24,000 pickles. They're saving pickles in Erewhon. So forget about option B, which is a budget of 34,000 pickles.

And the last thing I'd like to do is to clarify item two on the list, which is the PhD research by the University of Minnesota. What exactly does this mean? Well it's useful if you try and draw a graph. If you plot species against area, you end up with a regression that is called a **Species Richness Graph**. And, basically, the argument is that in the valley, for every hectare you add in area, you are likely to save an extra five species. So the slope of this line over here is five species per hectare.

But remember there are only 350 species in the valley. So when you reach 350, you max out at an area of 70 hectares. And then the number of species you would save if you bought extra land in the valley, in theory at least, would be none extra. You'd have that maximum of 350.

In the highland, you've got a lower slope. And that maxes out at 240 species. And since the slope is three species per hectare, you would max out in an area of 80 hectares.

So all the information in item two is really just the Species Richness Curve.

I'd like you to see how many species you can save. I'd like you to take about 20 minutes. Don't spend more than 25 minutes on it. Try to solve the problem in any way you think fit, and let's see how many of those species you can end up saving.

PAUSE the video: Work on the problem. How many species can you save?

Well, how many did you manage to save? Are you sure that that is the best that you can do? Can you guarantee that that's the best you can do? Are you sure you haven't violated any of the conditions?

Well, what I'm going to do is show you a paradigm for solving this type of problem. And I'm going to try and show you how some heuristics might have led you to this paradigm. But, in fact, in doing so, it reminds me of the most irritating thing that teachers used to do in high school. Where they sort of suggested you might have come up with the answer by yourself, because, in fact, I know what I'm heading towards, and you didn't when you were dealing with this problem. But let's see how it goes.

There's a lot of information in this problem. So the first heuristic I would probably use is to try and introduce some kind of notation to bring some order into the problem. Well let's think about a notation. What are we trying to find here? We're trying to decide how many hectares to buy in the highland and how many to buy in the valley.

So what we could do is use the letter V to represent the number of hectares we buy in the valley and H, the number of hectares we buy in the highland. This goes right back to high school, where if you had a problem with two unknowns, you call them X and Y. We're calling them V and H.

If you do that, we know how to calculate the number of species you save. Because for every hectare you buy in the valley, you're going to save five species, and for every hectare in the highland, you're going to save an additional three. So the total equation for species saved is 5 times V plus 3 times H. And this is something that we want to maximize, but we have to maximize it subject to certain conditions. Let's see if we can write out those conditions in terms of V and H.

The first set of conditions are what we call **Edge Effects**. And it tells us that V cannot be less than 15 hectares. We could write that as the inequality, V is greater than or equal to 15, and let's call that Condition 1. Similarly, H has got to be greater than or equal to 25. We can write that down as an inequality, and we can call that Condition 2.

The next issue is concerned with **politics**. And here, the story is that you are not to buy more than twice as many hectares in one area as you bought in the other area. And if you want to write that down, it tells you that V can't be bigger than twice H, or V is less than or equal to 2H, which we're going to label as Condition 3. But vice versa, H can't be bigger than 2V. So H must be less than or equal to 2V. And let's call that Condition 4.

Now we move on to the **social conditions**. And we have this concept of a social upheaval index, which is calculated by taking V and multiplying it by 5 and adding in H multiplied by 2. So your social upheaval index is 5 times V plus 2 times H. And that has to be less than or equal to 400. We'll call that Condition 5.

And then we have our **budget condition**. Each hectare in the valley costs 300 pickles. So if we buy V hectares, the total cost is 300 times V. And, similarly, the cost of buying H hectares in the highland is 200 times H. So 300 times V plus 200 times H is our total cost. And that has to be less than or equal to 24,000. And that is Condition number 6.

There are actually **two more conditions** because if you remember the Species Richness Curves, there's no point in buying more than 70 hectares in the valley and 80 hectares in the highland because you're going to max out on the species that you save. So V has to be less than or equal to 350 over 5, which is 70. Call that Condition 7. And, finally, H has got to be less than or equal to 240 over 3, which is 80. Call that Condition 8.

And those are our eight conditions. Well we've now summarized the problem, but this doesn't solve it for us. Again, if you remember back to word problems in high school, if there were two unknowns, there was usually information that led to two equations. And then you figured out how to solve those two equations, and you got your answer. Here we have two unknowns, but we don't have any equations. We have eight inequalities, which are conditions. And then we have an expression for the number of species that we save, and we have to max out on that. What are we going to do?

Well another heuristic is that sometimes, it's useful to put something into a diagram or to graph it. And, again, if you think back to high school, if you had two unknowns, you would probably graph the two equations to understand where the intersection was and to find your answer.

So as a bit of a long shot, we're going to try and take the information we have here and put it on a graph. What kind of graph? Well, we could plot V against H or H against V. Let's have a look at what that might look like.

What I have here is a graph of V versus H, and, obviously, we're only interested in positive values of V and of H. If I were to pick any point on the graph, suppose I pick a point there, then it has a V value and an H value. If I were to take those values and feed them into the equation for species saved, then associated with that point is a certain number of species. What we are looking for here is the point that saves the maximum number of species, but it has to be subject to our conditions. So if I just pick any point on the graph, I want to know whether or not that point satisfies all the conditions we have to meet. So it looks as though it might make sense to try and graph the conditions, somehow, on this diagram.

Well, if we take our first condition, the edge effect, V has to be greater than or equal to 15. We know how to draw the graph V equals 15. It's just a straight line parallel to the H axis that passes through V equals 15. Let's draw it. Well, that's the line V equals 15. We're interested in V is greater than or equal to 15. Well, I could just put a little arrow on it saying, "That side of the line," and say, "That is Condition 1."

I can actually do better than the arrow. I could hatch out everything below the line and say, "Any point below the line is unacceptable." That looks quite neat. Let's keep going.

Our next edge effect is that H has to be greater than or equal to 25. Well here's H equals 25. Let's draw the line. And H greater than or equal to 25 would be that side of the line, which is now Condition 2. And, again, I can hatch out everything to the left of the line.

Instead of moving on to Condition 3 now, what I'd like to do is jump ahead to Condition 7. Because there are upper limits on V and H when we've maxed out on all species. Remember the graph we drew of the species richness? V maxes out at 350 over 5, which is 70. So I'm going to draw the line V equals 70. And I have to be below that line. And that was condition 7.

And then H has to be less than or equal to 240 over 3, which is 80. So Condition 8 is to the left of that line. And the beauty of jumping ahead to 7 and 8 is we now know that our answer has to lie inside this rectangle. It looks like we are on a roll. Let's fill in the other conditions.

So let's go back to Condition number 3. Three is V is less than or equal to 2H. Well that passes through the origin. And when H is 25, 2H is 50. So that is going to be a graph that looks like this. And that was Condition number 3. And we have to be below the line. Is it obvious that we have to be below the line? If you're not sure which side of the line you need to be, it's very easy to figure that out. So choose a point below the line. Choose H equals 40 and V equals 0. If H is 40 and V is 0, is V less than 2H? Yes. So, obviously, below the line works.

Then we want to do Condition number 4, H is less than or equal to 2V. We want to draw the line H equals 2V, which is the same as the line V is equal to a half H. So, for example, when H is 40, V is going to be 20. So that is going to be a line that looks like this. Which side of the line do we need to be? Well, let's do our test. If H is 40 and V is 0, is H less than twice 0? No. So we have to be above the line. That's Condition 4. And notice we've now reduced the space in which our answer has to lie. That area goes out, and that area goes out.

We only have two conditions left to put in. The next is Condition number 5, which is the social upheaval. And then we have the graph 5V plus 2H. And 5V plus 2H has to be equal to 400.

A good way to graph a line like that is to see where it cuts the axes. So if I were to put H equal to 0, 5V would be 400. V would be 80. So it has to pass through that point, V equals 80. If I put V equal to 0, H is 200, which is right off my graph. So let's say what happens if H is 100? Then 2H is 200, 5V has to be equal to 400 minus 200, which is 200. V has to be equal to 40. So if we take the point H equals 100 and V equals 40, then my social upheaval index line is going to pass through there. So I'm going to join the point V equals 80 with that point, and that's Condition 5. And you can confirm that we have to be below that line. And we've got a little more to hatch out.

Finally, we put in our budget. This is Condition number 6. 300V plus 200H equals 24,000 is what we've got to draw. Again, if we put H equal to 0, V will be 80. So it's also going to pass through the point V equals 80. If we put V equal to 0, H will be 120. So it joins up with this point over here. Let's draw the line. And that is my Condition number 6. We have to be below the line, and we can hatch out these points.

And that is as far as we can get with our conditions. But notice that we've at least reached the conclusion that our answer has to lie somewhere inside this polygon, okay?

Suppose we'd put in all the conditions, and the whole graph were hatched out. Suppose this was hatched out as well. Well that would tell you the useful information that the conditions are so stringent that there isn't a solution to the problem.

We've still got a lot of points inside this polygon. Which one's the right answer?

Suppose I were to pick a point at random inside the polygon. Suppose I were to pick that point. Then, as I said earlier, you can calculate 5V plus 3H. By increasing either V or H or both, I can end up saving more species. So if I were to move to the right or up or any direction like that, I would end up saving more species.

So I could ask the question, "What stops me from saving more and more species?" And the answer is, "Eventually, I'm going to hit one of the boundaries of the polygon."

Can you see that I'm sort of suggesting that there's a theorem that says, "The answer doesn't lie inside the polygon. It actually lies somewhere on the boundary." Well you can actually prove a theorem like that. So now I know that the answer has to lie somewhere on the boundary of this polygon.

Well, again, suppose I were to pick any point on any one of the boundary lines. Suppose I were to pick a point here, and I were to ask, "Can I do better?" The answer is, "If I can increase V and H, I can do better. Well, I could move up that line." What stops me from moving up that line as far as I like? I'm going to hit another line.

So I'm now suggesting my theorem isn't that that the answer lies on the boundary. My theorem is that the answer has to lie at one of the intersections of the boundary line. And we only have one, two, three, four, five intersections. One of those is our answer. So we've now reduced the whole problem to five possible points.

Remember back in high school, the answer lay at the intersection between two lines? It's true here. Our problem is finding which two lines.

Well, if you look at this diagram, if I've drawn it well, it's pretty obvious that the answer can't be either of these two points. And it almost certainly isn't that point because that point has a high V and H. So the answer either lies there or there. I can calculate the exact coordinates of that point and the exact coordinates of that point, and I can feed those coordinates into my expression for the total species saved, 5V plus 3H. And if I do that, I'll find out that this point gives me the better answer. This point is the point at which H equals 30, V equals 60, and I save 390 species.

Is that the answer you got? If you did better than that, I bet you made a mistake. You didn't satisfy all the conditions. If you did worse than that, well, now you know that you can do better.

If you have drawn this graph, and you've drawn it accurately, there is, in fact, a graphical way to find a solution point. My graph is very messy. So I'm not going to draw it for you, but I'm going to tell you what you might do.

Suppose you go back to the equation for species saved, 5V plus 3H. Ask yourself the hypothetical question, "Can I save 150," for example, "species"? If 5V plus 3H equals 150, that's a straight line on your graph. Draw it. Draw it in a different color if you can. It's pretty obvious once you've drawn it that you can save more species.

So try to draw the line 5V plus 3H equals 300. What's the relationship between the two lines? Well, obviously, they're parallel. So if you move a straight edge parallel to those two lines up and up, you will pass through various points until you hit the one point that is the very last point on the polygon. That's the answer. And, as I said, if you've drawn it properly, you're going to come up with the answer of 60 hectares in the valley, 30 in the highland, and you save 390 species. Guaranteed!

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