De Rham Cohomology

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1 Introduction

One of the main goals of algebraic topology is finding homeomorphism invariants: certain structures that don't change under such transformation. These invariants allow us to tell different spaces apart. In point set topology, we were introduced certain topological properties such as compactness, connectedness and Hausdorffness. These are all homeomorphism invariants, but are too weak to differentiate between different spaces and give a satisfying characterization.

In algebraic topology, we encountered the fundamental group $\pi_1(X, p)$ for a space X, which consists of all the loops on X that cannot be deformed into each other. It is, in fact, sufficient to classify the closed orientated 2-dimensional surfaces, but insufficient for higher dimensions. This can be generalized to higher homotopy groups $\pi_k(X, p)$, but the drawback is that they are very difficult to compute. In this paper, we will introduce a homeomorphism (and homotopy) invariant that's powerful but relatively easy to compute, called the de Rham cohomology. Roughly speaking, it characterizes the topological properties of X by measuring the extent to which the fundamental theorem of calculus fails on X.

Towards this end, we introduce the theory of differential forms in Section 2. We understand them as dual objects to vector fields: just as a vector field assigns to each point a tangent vector in $\mathbb{R}^n$, a differential k-form assigns to each point a k-covector. The calculus of differential forms generalizes the vector calculus on Euclidean spaces to manifolds. Section 3 gives a formal introduction to de Rham cohomology, building up to the Mayer-Vietoris sequence—the main algebraic tool to compute cohomologies—and homotopy invariance of the de Rham cohomology. Section 4 introduces some applications of de rham cohomology, such as the cohomology of the sphere, Brower's Fixed Point Theorem, and the Hopf Invariant.

We will assume knowledge of linear algebra, point-set topology, the basics of algebraic topology (i.e. the contents of this course up to homotopy), and familiarity with groups. Knowledge of manifolds is helpful but not required; for the majority of this paper, we will treat manifolds as open sets in $\mathbb{R}^n$.

2 Basics in Differential Geometry

Definition 2.1 (C^∞ functions)**.** A function $f : U \to \mathbb{R}$ on a open set U in $\mathbb{R}^n$ is said to be *smooth*, or C^∞, if its partial derivatives $\partial^j f/\partial x^{i_1}...\partial x^{i_1}$ of all orders exist, and are continuous at every point $p \in U$.

2.1 Tangent Vectors in $\mathbb{R}^n$

In calculus, we treat a vector v at a point $p \in \mathbb{R}^n$ as a collection of numbers $v = \langle v_1, v_2, ..., v_n\rangle$, or geometrically, as an arrow emancipating from p. However, these characterizations are hard to generalize to more abstract objects such as manifolds, which are not embedded in a Euclidean space. Instead, we will think about vectors as certain operators on functions, which are much easier to work with.

Let U be an open set in $\mathbb{R}^n$, with the standard Cartesian coordinates $(x^1, ..., x^n)$.

Definition 2.2 (Tangent space)**.** The tangent space to U at p is defined as $T_p(U) = \{(p, v) : v \in \mathbb{R}^n\}$.

There is a natural correspondence between $T_p(U)$ and U, given by $(p, v) \mapsto v$. When the choice of the base point is clear, we will simply refer to (p, v) as v.

For each $v = \langle v_1, v_2, ..., v_n\rangle \in T_p(U)$ and each C^∞ function f on U, we denote $D_v f$ as the *directional derivative* of f in the direction of v at p. By the chain rule,

$$D_v f = \sum_i v^i \frac{\partial f}{\partial x^i}(p).$$

We write $D_v = \sum v^i \frac{\partial}{\partial x^i}|_p$ as the map that sends f to $D_v f$. The identification $v \mapsto D_v$ allows us to treat tangent vectors as operators on functions. It is clear from the identification that $\{\frac{\partial}{\partial x^i}|_p\}$ is a basis for $T_p(\mathbb{R}^n)$.

Definition 2.3. A *vector field* X assigns to each point $p \in U$ a tangent vector $X_p \in T_p(U)$.

Explicitly, a vector field can be written as $X = \sum_i a^i \frac{\partial}{\partial x^i}$, where a^i are C^∞ functions on U.

2.2 Wedge product and Multicovectors

Moving on from tangent vectors, we take the dual point of view and study the linear functions on a tangent space. This turns out to be more fruitful, as there are much more we can do with linear functions: we can add them, multiply them, compose them, and scalar multiply them. We are especially interested in alternating multilinear functions, or *mulitcovectors*, as they enable us to generalize familar notions in vector calculus.

Definition 2.4 (Multilinear functions)**.** Let V^k denote the Cartesian product of k copies of a real vector space V. A function $V^k \to \mathbb{R}$ is *k-linear* if it is linear in each of its k arguments:

$$f(..., av + bw, ...) = af(..., v, ...) + bf(..., w, ...).$$

The symmetric group S_k acts on the group of k-linear functions in the following way: If f is a k-linear function and σ is a permutation in S_k, σf is a new k-linear function defined by

$$(\sigma f)(v_1, ..., v_k) = f(v_{\sigma(1)}, ..., v_{\sigma(k)}).$$

In other words, the action of σ permutes the indices of the vector space V.

Definition 2.5. We say a k-linear function f is *alternating* if $\sigma f = (\operatorname{sgn}\sigma)f$ for all $\sigma \in S_k$. An alternating k-linear function is also known as a *k-covector*, or a *multicovector of degree k*. The space of all alternating k-linear function on V is denoted as $A_k(V)$. When $k = 0$, we define a *0-covector* to be constant. A 1-covector is simply a *covector*.

Example 2.6.
(i) The cross product $v \times w$ on $\mathbb{R}^3$ is a 2-covector.
(ii) The determinant $f(v_1, ..., v_n) = \det(v_1, ..., v_n)$ on $\mathbb{R}^n$ is a n-covector.

If f is an alternating k-linear function, and g is an alternating l-linear function, we would like to have a product that is $(k + l)$-linear and also alternating. This motivates the following

Definition 2.7 (Wedge product)**.** For $f \in A_k(V)$ and $g \in A_l(V)$, the *wedge product* of f and g is given by

$$f \wedge g(v_1, ...v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma) f(v_{\sigma(1)}, ...v_{\sigma(k)}) g(v_{\sigma(k+1)}, ..., v_{\sigma(k+l)}).$$

Here $\operatorname{sgn}(\sigma)$ denotes the signature of the σ. When $k = 0$, the function $f \in A_0(V)$ is simply a constant c. In this case, the wedge product $c \wedge g$ is scalar multiplication.

Remark 2.8. It is not hard to check that $f \wedge g$ is indeed an alternating $(k+l)$-linear function; we leave this to the readers.

Remark 2.9. The coefficient $\frac{1}{k!l!}$ compensates for repetitions in the sum: for every $\sigma \in S_{k+l}$, there are $k!$ permutations τ that do not change the arguments of g. These permutations contribute the same term in the sum, since

$$(\operatorname{sgn}\sigma\tau)f(v_{\sigma\tau(1)},...,v_{\sigma\tau(k)}) = (\operatorname{sgn}\sigma\tau)(\operatorname{sgn}\tau)f(v_{\sigma(1)},...,v_{\sigma(k)}) = (\operatorname{sgn}\sigma)f(v_{\sigma(1)},...,v_{\sigma(k)}).$$

Similarly, there are $l!$ permutations that do not change the arguments of f. We divide $k!l!$ to account for these repetitions.

Example 2.10 (Wedge Product and Cross Product)**.** Let V be a 3-dimensional vector space with basis e_1, e_2, e_3, and dual basis $\alpha^1, \alpha^2, \alpha^3$. At each point, we can associate a 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V with a vector $v_\alpha = (a_1, a_2, a_3) \in \mathbb{R}^3$; a 2-covector

$$\gamma = c_1\alpha^2 \wedge \alpha^3 + c_2\alpha^3 \wedge \alpha^1 + c_3\alpha^1 \wedge \alpha^2$$

with $v_\gamma = (c_1, c_2, c_3) \in \mathbb{R}^3$.

Let $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$, $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$. Then we have

$$\alpha \wedge \beta = (a_2b_3 - a_3b_2)\alpha^2 \wedge \alpha^3 + (a_3b_1 - a_1b_3)\alpha^3 \wedge \alpha^1 + (a_1b_2 - a_2b_1)\alpha^2 \wedge \alpha^3.$$

This corresponds to the cross product of vectors in $\mathbb{R}^3$, i.e. $v_{\alpha\wedge\beta} = v_\alpha \times v_\beta$. Thus, the wedge product is a generalization of the cross product from $\mathbb{R}^3$ to $\mathbb{R}^n$.

Proposition 2.11 (Anticommutativity)**.** *If $f \in A_k(V)$ and $g \in A_l(V)$, then $f \wedge g = (-1)^{kl} g \wedge f$.*

Proof. Define permutation $\tau \in S_{k+l}$, where $\tau(i) = \begin{cases} i+k, & \text{if } i \le l \\ i-l, & \text{if } i > l \end{cases}$. It follows that

$$\begin{aligned} f \wedge g &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma) f(v_{\sigma(1)}, ...v_{\sigma(k)}) g(v_{\sigma(k+1)}, ..., v_{\sigma(k+l)}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma) f(v_{\sigma\tau(l+1)}, ...v_{\sigma\tau(l+k)}) g(v_{\sigma\tau(1)}, ..., v_{\sigma\tau(l)}) \\ &= \frac{\operatorname{sgn}(\tau)}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma\tau) f(v_{\sigma\tau(l+1)}, ...v_{\sigma\tau(l+k)}) g(v_{\sigma\tau(1)}, ..., v_{\sigma\tau(l)}) \\ &= (\operatorname{sgn}\tau) g \wedge f. \end{aligned}$$

The last equality follows from the fact that when σ runs through all of S_{k+l}, so does $\sigma\tau$. It remains to check that $\operatorname{sgn}\tau = (-1)^{kl}$, which will be left as an exercise. ■

Corollary 2.12. *For $f \in A_k(V)$, $f \wedge f = 0$.*

Given a vector space V, denote its basis as e_1,..., e_n, and its dual basis as $\alpha^1, ..., \alpha^n$. Using the multi-index notation $I = (i_1, ...i_k)$, we write e^I for $(e_{i_1}, ..., e_{i_k})$ and α^I for $(\alpha^{i_1} \wedge ... \wedge \alpha^{i_k})$. For the rest of this section, we seek to find a basis for $A_k(V)$, the space of k-covectors on V.

Lemma 2.13 (Wedge product of covectors)**.** *Let $\alpha^1, ..., \alpha^k$ be covectors on a vector space V, and $v_1, ..., v_k \in V$. It follows from a straightforward computation that*

$$(\alpha^1 \wedge ... \wedge \alpha^k)(v_1, ..., v_k) = \det[\alpha^i(v_j)]$$

. The right-hand side of the equation denotes the determinant of the matrix with $\alpha^i(v_j)$ as its (i, j) argument.

Corollary 2.14. *Let V be a n-dimensional vector space with basis $e_1, ..., e_n$ and dual basis $\alpha^1, ..., \alpha^n$. If $I = (i_1 < ... < i_k)$ and $J = (j_1 < ... < j_k)$ are multi-indices of length k, then*

$$\alpha^I(e_J) = \delta^I_J = \begin{cases} 1, & \text{if } I = J, \\ 0, & \text{if } I \neq J. \end{cases}$$

Proof. By Lemma 2.13, we have

$$\alpha^I(e_J) = \det[\alpha^i(e_j)]_{i \in I, j \in J}.$$

If $I = J$, then $[\alpha^i(e_j)]$ is the identity matrix, which has determinant 1. Otherwise, if $I \neq J$, we consider the smallest l for which $i_l \neq j_l$. Without loss of generality, assume that $i_l < j_l$. Observe that $i_l \neq j_k$ for $k < l$, because $j_k = i_k$ and I is strictly ascending. Furthermore, $i_l \neq j_k$ for $k \geq l$, since $j_l > i_l$ and J is strictly ascending. Thus, the l-th row of the matrix $\alpha^I(e_J)$ is all 0s, which implies its determinant is 0. ■

Proposition 2.15. *The alternating k-linear functions α^I, $I = (i_1 < ... < i_k)$, form a basis for $A_k(V)$.*

Proof. First, we will show linear independence. Suppose $\sum_I c_I \alpha^I = 0$. Applying both sides to e_J, we get

$$0 = \sum_I c_I \alpha^I(e_J) = \sum_I c_I \delta^I_J = c_J.$$

To show that the α^I span $A_k(V)$, we claim that for any $f \in A_k(V)$,

$$f = \sum_I f(e_I)\alpha^I.$$

This is clear if we again apply e_J to both sides. Thus, the α_I form a basis for $A_k(V)$. ■

2.3 Differential forms

In this section, we will introduce the theory of differential forms. They can be viewed as a generalization of covector fields, which are, after all, differential 1-forms. Differential forms provide us with a framework to generalize notions in vector calculus such as gradient, divergence, and curl. More importantly, they turn out to be objects that can be integrated over manifolds. We therefore give meaning to dx, dy, dz—what was merely a notation in vector calculus.

Definition 2.16 (Differential 1-form)**.** A *differential 1-form*, a *1-form*, or a *covector field* on an open subset U of $\mathbb{R}^n$ is a function ω that assigns to each point p in U a covector $\omega_p \in T^*_p(\mathbb{R}^n)$,

$$\omega : U \to \bigcup_{p \in U} T^*_p(\mathbb{R}^n),$$
$$p \mapsto \omega_p \in T^*_p(\mathbb{R}^n).$$

A C^∞ function $f : U \to \mathbb{R}$ induces a 1-form df, called the *differential of f*, defined pointwise by

$$(df)_p(X_p) = X_p f, \quad \forall p \in U.$$

Intuitively, one may think of $(df)_p$ as an infinitesimal change of f at point p.

Let $x^1, ..., x^n$ be the standard coordinates on $\mathbb{R}^n$. The set of tangent vectors $\{\frac{\partial}{\partial x^1}|_p, ..., \frac{\partial}{\partial x^n}|_p\}$ is a basis for $T_p(\mathbb{R}^n)$. Note that

$$(dx^i)_p\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \frac{\partial}{\partial x^j}\Big|_p(x^i) = \delta^i_j.$$

Thus, the set $\{(dx^1)_p, ..., (dx^n)_p\}$ is a basis for the dual tangent space $T^*_p(\mathbb{R}^n)$.

Definition 2.17 (Differential k-form)**.** A *differential k-form* or a *k-form* on an open subset U of $\mathbb{R}^n$ assigns each point p in U a k-covector $\omega_p \in A_k(T_p(\mathbb{R}^n))$.

By Proposition 2.15, the set $\{dx^I_p\}_{I=(1\leq i_1<...<i_k\leq n)}$ forms a basis for $A_k(T_p(\mathbb{R}^n))$. Thus, ω_p can be written as a linear combination

$$\omega_p = \sum a_I(p)dx^I_p, \;\; 1 \leq i_1 \leq ... \leq i_k \leq n,$$

where a_I are functions from U to $\mathbb{R}$. Globally,

$$\omega = \sum_I a_I dx^I, \;\; 1 \leq i_1 \leq ... \leq i_k \leq n.$$

Here, ω is said to be C^∞ if all the a_I are C^∞.

One should not let the formal definition of differential forms obscure its geometric intuition: a k-form can be thought of as a measuring device for orientated k-dimensional volume elements. For $k = 1$, a good analogy is that if the vector field is a velocity field, then a 1-form is a "speedometer field." To make this more concrete, let e_i be the Cartesian basis elements of $\mathbb{R}^3$, dx^i be the coordinate 1-forms, and a_i be smooth functions. A 1-form on $\mathbb{R}^3$ looks like

$$\omega = a_1dx^1 + a_2dx^2 + a_3dx^3.$$

If $X = X^1e_1 + X^2e_2 + X^3e_3$ is a vector field on $\mathbb{R}^3$, then

$$\omega(X) = a_1X^1 + a_2X^2 + a_3X^3$$

may be viewed as a measurement: it is the sum of a_i times the i-th component of X.

For a 2-form $\omega = a_1dx^2 \wedge dx^3 + a_2dx^3 \wedge dx^1 + a_3dx^1 \wedge dx^2$ and a bivector field $X = (V, W)$, we have

$$\omega(X) = a_1dx^2 \wedge dx^3(V, W) + a_2dx^3 \wedge dx^1(V, W) + a_3dx^1 \wedge dx^2(V, W).$$

To illustrate what this looks like, we expand one component of this sum, say $a_3dx^1 \wedge dx^2(V, W)$:

$$a_3dx^1 \wedge dx^2(V, W) = a_3\Big(dx^1(V)dx^2(W) - dx^1(W)dx^2(V)\Big) = a_3(V^1W^2 - W^1V^2). \tag{1}$$

At each point, V and W span a parallelopiped in $\mathbb{R}^3$. The right hand side of (1) is exactly a_3 times the projection area of this parallelopiped onto the (x^1, x^2) plane (see Figure 1).

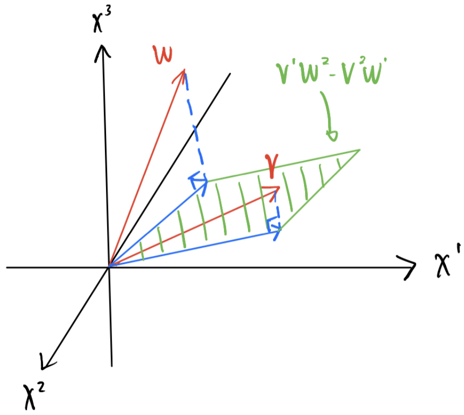

Figure 1: The x^1,x^2 component of $\omega(X)$

For a general k, the differential k-form at each point is a measuring device that takes in an k-dimensional parallelopiped (technically, it takes in k tangent vectors, which one should think of as spanning a k-dimensional parallelopiped), and spits out a number proportional to its hypervolume. The formal definition, "a multilinear alternating function on k tangent vectors," says the same thing: multilinear and alternating means that it spits out a number proportional to the length/area/volume/hypervolume. Note that the determinant, which is used to compute the volume of a parallelopiped (and its higher and lower dimensional analogues), has the same properties. In a sense, covectors are generalizations of the determinant, as illustrated in Lemma 2.13.

Next, we introduce an important operation on differential forms, called the exterior derivative, which generalizes the divergence, gradient and curl operators in the vector calculus of $\mathbb{R}^3$.

Definition 2.18 (Exterior derivative)**.** The *exterior derivative* of a C^∞ k-form $\omega = \sum a_I dx^I$ is

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \Big(\sum_j \frac{\partial a_I}{\partial x^j} dx^j \Big) \wedge dx^I \in \Omega^{k+1}(U).$$

If $k = 0$, the exterior derivative of $\omega : U \to \mathbb{R}$ is its differential $d\omega$.

This formulism is nice, but we will mostly work with the exterior derivative in terms of its axioms.

Definition 2.19 (Exterior derivative)**.** The *exterior derivative* is the unique linear map $d : \Omega^*(U) \to \Omega^*(U)$ that satisfies the following properties:
(i) For a k-form ω and a l-form τ, $d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^k \omega \wedge d\tau$.
(ii) $d^2 = 0$.
(iii) If $f \in C^\infty(U)$, then $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$.

Example 2.20 (Exterior derivative and div, grad, curl)**.** On $\mathbb{R}^3$, we can identify a function f with a 0-form f or a 3-form $f dxdydz$, as they are all 1-dimensional. Similarly, we can identify a vector field $X = (f_1, f_2, f_3)$ with a 1-form $f_1 dx + f_2 dy + f_3 dz$ or a 2-form

$$f_1 dydz + f_2 dzdx + f_3 dxdy,$$

as they are all 3-dimensional. Applying the exterior derivative on functions, we get

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

On 1-forms,

$$d(f_1dx + f_2dy + f_3dz) = \Big(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\Big)dydz + \Big(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\Big)dzdx + \Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big)dxdy.$$

On 2-forms,

$$d(f_1dydz + f_2dzdx + f_3dxdy) = \Big(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\Big).$$

Thus, we have the following correspondence

$$\begin{aligned} d(0-\text{forms}) &\longleftrightarrow \text{gradient},\\ d(1-\text{forms}) &\longleftrightarrow \text{curl},\\ d(2-\text{forms}) &\longleftrightarrow \text{divergence}. \end{aligned}$$

Definition 2.21. A differential k-form ω is *closed* if $d\omega = 0$, and *exact* if $\omega = d\tau$ for some $(k-1)$-form τ.

2.4 Pullbacks and Pushforwards

Definition 2.22 (Pullback of smooth functions)**.** Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth function. For any smooth function $g : \mathbb{R}^n \to \mathbb{R}$, the *pullback of* g *by* f is defined as $f^*(g) : \mathbb{R}^m \to \mathbb{R}$, where$f^*(g) = g \circ f$.

By composition with f, a function on the codomain of f is "pulled back" to a function on the domain of f. Tangent vectors are operators on functions, so while functions get pulled back, tangent vectors are sent forward. Indeed, we can see this in the following

Definition 2.23 (Pushforward of tangent vectors)**.** Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth function. At each point $p \in \mathbb{R}^m$, f induces a linear map of tangent spaces, called the *pushforward of* f *at* p,

$$f_{*,p}(X_p)(g) = X_p(g \circ f)$$

If the context is clear, we write the pushforward simply as f_*. The readers can check that $f_{*,p} : T_p(\mathbb{R}^m) \to T_{f(p)}(\mathbb{R}^n)$ is indeed a linear map. Intuitively, one may think of f_* as a local linearization of f.

Differential forms are dual objects to vector fields. At each point, a k-form define a k-covector on the tangent space. Just as functions operate on a point, these covectors operate on tangent vectors. As such, just like functions, they can be pulled back.

Definition 2.24 (Pullback of k-forms)**.** If ω is a k-form on $\mathbb{R}^n$, then its *pullback* $f^*\omega$ is defined pointwise by $(f^*\omega)_p = f^*(\omega_p)$ for all $p \in \mathbb{R}^n$, where $f^*(\omega_p)$ is a k-covector at p in $\mathbb{R}^m$ given by

$$f^*(\omega_p)(v_1, ..., v_k) = \omega_p(f_{*,p}(v_1), ..., f_{*,p}(v_k)), \quad v_i \in T_p(\mathbb{R}^m).$$

Just like the pullback of functions, the pullback of a k-form can be viewed as a composition

$$T_p(\mathbb{R}^m) \times ... \times T_p(\mathbb{R}^m) \xrightarrow{F_* \times ... \times F_*} T_{f(p)}(\mathbb{R}^n) \times ... \times T_{f(p)}(\mathbb{R}^n) \xrightarrow{\omega_{F(p)}} \mathbb{R}.$$

Using the definition of pullbacks, one can check the following

Proposition 2.25. *For $c \in \mathbb{R}$, function $f : \mathbb{R}^m \to \mathbb{R}^n$, and differential forms $\omega, \omega_1, \omega_2$ on $\mathbb{R}^n$, we have:*
(i) $f^*(c\omega) = cf^*\omega$,
(ii) $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$,
(iii) $f^*(\omega_1 \wedge \omega_2) = f^*(\omega_1) \wedge f^*(\omega_2)$,
(iv) $f^*(d\omega) = df^*(\omega)$.

Proposition 2.26. *Let $f : U \to V$ and $g : V \to W$ be smooth maps on real vector spaces. One can check by definition that the pullback of forms satisfy $(g \circ f)^* = f^* \circ g^*$.*

3 De Rham Cohomology

3.1 De Rham Cohomology

We will start off this section with some motivation from physics. Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a smooth vector field representing a force over $U \subseteq \mathbb{R}^2$, and let C be a parametrized curve from p to q. The work done by the force along C is given by $\int_C Pdx + Qdy$. If $\mathbf{F} = \nabla f = \langle f_x, f_y\rangle$, the line integral is easy to compute by Stokes theorem:

$$\int_C Pdx + Qdy = \int_C f_x dx + f_y dy = \int_C df = f(q) - f(p). \tag{2}$$

A necessary condition for $\mathbf{F}$ to be a gradient is that $P_y = f_{xy} = f_{yx} = Q_x$. One naturally wonders if the converse is true: suppose $P_y = Q_x$, does $\mathbf{F} = \nabla f$ for some function f on U?

As seen from Example 2.10, there is a natural correspondence between vector fields and differential 1-forms on U:

$$\begin{aligned} \mathbf{F} = \langle P, Q\rangle &\longleftrightarrow \omega = Pdx + Qdy, \\ \nabla f = \langle f_x, f_y\rangle &\longleftrightarrow df = f_x dx + f_y dy, \\ Q_x - P_y = 0 &\longleftrightarrow d\omega = (Q_x - P_y)dx \wedge dy = 0. \end{aligned}$$

The question then becomes: if the one-form ω is closed, is it exact? The answer to this question depends on the topology of U.

Let U be a open set in $\mathbb{R}^n$. Define $Z^k(U)$ as the vector space of all closed k-forms of U, and $B^k(U)$ as the vector space of all exact k-forms of U.

Definition 3.1. The quotient space $H^k(U) := Z^k(U)/B^k(U)$ is called the *de Rham cohomology* of U.

We call the equivalence class of a closed form ω its *cohomology class*. If ω and ω' are in the same cohomology class, we say they're *cohomologous*. The de Rham cohomology measures the extent to which closed k-forms fail to be exact.

Proposition 3.2. *If U has r connected components, then $H^0(U) = \mathbb{R}^r$.*

Proof. There are no exact 0-forms on U, so $H^0(U)$ is simply the set of closed 0-forms. Let f be a closed 0-form, i.e. f is a C^∞ function such that

$$df = \sum \frac{\partial f}{\partial x^i} dx^i = 0.$$

This means that $\frac{\partial f}{\partial x^i} = 0$ for all i, so f is locally constant. From point-set topology, we know that a locally constant function on U is constant on each of its connected components. If U has r connected components, then the locally constant functions can be determined by r real numbers. Thus, $H^0(U) = \mathbb{R}^r$. ■

Proposition 3.3. *If U has dimension m, then its de Rham cohomology in degree k is $H^k(U) = 0$, if $k > m$.*

Proof. Suppose ω is a k-form on U. At each point $p \in U$, ω_p is a k-covector on $T_p(U)$, which has dimension m. If we write ω_p explicitly as a combination of $dx^{i_1} \wedge ... \wedge dx^{i_k}$, then at least two of the factors are the same. By Proposition 2.11 and Corollary 2.12, this implies that $\omega_p = 0$. So every k-form on U is 0. ■

Example 3.4 (de Rham cohomology of the real line)**.** From Proposition 3.2, we know that $H^0(\mathbb{R}) = \mathbb{R}$. From Proposition 3.3, $H^k(\mathbb{R}) = 0$ for $k \geq 2$. To find $H^1(\mathbb{R})$, observe that every 1-form is closed. Moreover, a 1-form $f(x)dx$ is exact if and only if $f(x)dx = dg = g'(x)dx$ for some C^∞ function g on $\mathbb{R}$. We can simply let g be the antiderivative of f, i.e. $g = \int_0^x f(t)dt$. This shows that every 1-form is exact. Therefore,

$$H^k(\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{if } k \geq 1. \end{cases}$$

Define the vector space $\Omega^*(U) = \bigoplus_{k=1}^n \Omega^k(U)$. The exterior derivative extends to $d : \Omega^*(U) \to \Omega^*(U)$. Together they form a chain of differential forms called the *de Rham complex of* U, denoted by $(\Omega^*(U), d)$:

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} ..., \quad d \circ d = 0.$$

The kernel of d are the closed forms, and the image of d are the exact forms. Note that the de Rham complex *always* exist for any open set U in $\mathbb{R}^n$. One may regard it as a free set of differential equations whose solutions are the closed forms: finding a closed form $fdx + gdy$ on $\mathbb{R}^2$, for instance, is equivalent to solving the differential equation $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0$.

3.2 The Mayer-Vietoris Sequence

As seen from Example 3.4, computations of the de Rham cohomology often amounts to solving a system of differential equations, which is usually difficult to do directly. This section introduces a powerful computational tool for the de Rham cohomology (and in fact any cohomology group) called the Mayer-Vietoris sequence. Given an open set X in $\mathbb{R}^n$, we can use the Mayer-Vietoris sequence to link the de Rham cohomology of X to the de Rham cohomology of its open covers, which we can choose to be readily computable.

This section involves a lot of what is commonly called *diagram chasing*. To get a good understanding of the Mayer-Vietoris sequence, the readers are encouraged to trace the proof along the diagrams.

Definition 3.5. Let $\{A^k\}$ be a collection of real vector spaces, and $f_k : A^k \to A^{k+1}$ be a linear map for each k. A sequence

$$A^0 \xrightarrow{f_0} A^1 \xrightarrow{f_1} ... \xrightarrow{f_{n-1}} A^n$$

is said to be *exact at* A^k if $\ker f_k = \operatorname{im} f_{k-1}$. It is an *exact sequence* if it's exact at every term except for the first and the last. In particular, a five-term exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called a *short exact sequence*.

There are some things one can immediately conclude from the definition of a short exact sequence: f must be injective, g must be surjective, and $C \simeq \frac{B}{\operatorname{im} f}$. This will be left to the readers.

Let X be an open set of $\mathbb{R}^n$, and U, V be an open cover of X. There is a natural sequence of inclusions as shown in Figure 2. Taking the pullback of these inclusions, we obtain a sequence of restrictions of k-forms, as shown in Figure 3.

$$\begin{tikzcd} & U \arrow[dr, "i_U"] & \\ U \cap V \arrow[ur, hook, "j_U"] \arrow[dr, hook, "j_V"'] & & X \\ & V \arrow[ur, hook, "i_V"'] & \end{tikzcd}$$

Figure 2: Sequence of Inclusions

$$\begin{tikzcd} & \Omega^k(U) \arrow[dr, "j_U^*"] & \\ \Omega^k(X) \arrow[ur, "i_U^*"] \arrow[dr, "i_V^*"'] & & \Omega^k(U \cap V) \\ & \Omega^k(V) \arrow[ur, "j_V^*"'] & \end{tikzcd}$$

Figure 3: Sequence of Restrictions

The restriction of a k-form from X to U and to V gives us a linear map

$$\begin{gathered} i^* : \Omega^k(X) \to \Omega^k(U) \bigoplus \Omega^k(V) \\ \sigma \mapsto (\sigma|_U, \sigma|_V). \end{gathered}$$

Additionally, define the difference map

$$\begin{aligned} j^* : \Omega^k(U) \bigoplus \Omega^k(V) &\to \Omega^k(U \cap V) \\ (\omega, \tau) &\mapsto \tau|_{U\cap V} - \sigma|_{U\cap V}. \end{aligned}$$

Then we get the following sequence, called the *Mayer-Vietoris sequence.*

$$0 \to \Omega^k(X) \xrightarrow{i^*} \Omega^k(U) \bigoplus \Omega^k(V) \xrightarrow{j^*} \Omega^k(U \cap V) \to 0.$$

Proposition 3.6. *The Mayer-Vietoris sequence is a short exact sequence.*

In order to prove this proposition, we must introduce a technical lemma showing the existence of a partition of unity, which is defined as follows.

Definition 3.7 (C^∞ partition of unity)**.** If $\{U_i\}_{i\in I}$ is a finite open cover of X, a C^∞ *partition of unity subordinate to* $\{U_i\}_{i\in I}$ is a collection of C^∞ functions $\{\rho_i : X \to \mathbb{R}\}_{i\in I}$ such that $\operatorname{supp} \rho_i \subset U_i$ and

$$\sum_i \rho_i = 1.$$

Lemma 3.8. *Let $\{U_i\}$ be a finite open cover of X. There exists a C^∞ partition of unity subordinate to $\{U_i\}$.*

For a proof of this lemma, see for example [6]. Now we are ready to prove Proposition 3.6.

Proof of Proposition 3.6. Exactness at $\Omega^k(X)$ and $\Omega^k(U) \bigoplus \Omega^k(V)$ follows directly from the definition of i^* and j^*. We will prove exactness at $\Omega^k(U \cap V)$, which amounts to showing that j^* is surjective.

First, consider the case when $k = 0$. Let f be a C^∞ function on $U \cap V$. We want to write f as the difference of a C^∞ function on V and a C^∞ function on U.

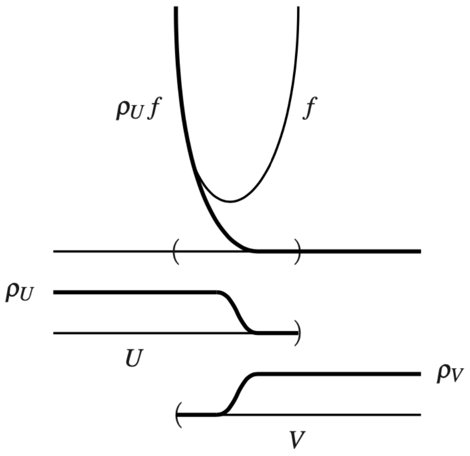

Figure 4: Writing f as a difference of a C^∞ function on V and a C^∞ function on U.

Since U, V is a finite cover of X, Lemma 3.8 tells us that there exists a partition of unity $\{\rho_U, \rho_V\}$ subordinate to U and V, respectively. Define $f_V(x) : V \to \mathbb{R}$, called *the extension by zero* of $\rho_U f$ from $U \cap V$ to V (see Figure 4), where

$$f_V(x) = \begin{cases} \rho_U(x) f(x) & \text{if } x \in U \cap V, \\ 0 & \text{if } x \in V - (U \cap V). \end{cases}$$

Similarly, define $f_U(x)$ to be the extension by zero of $\rho_V f$ from $U \cap V$ to U. One can check that f_U, f_V are smooth. Note that

$$j^*(-f_U, f_V) = f_V|_{U \cap V} + f_U|_{U \cap V} = \rho_U f + \rho_V f = f.$$

Thus, j^* is surjective.

We can apply the same procedure for differential k-forms. For a k-form ω on $U \cap V$, let ω_U, ω_V be the extension by zero from $U \cap V$ to U and to V, respectively. Then

$$j^*(-\omega_U, \omega_V) = \rho_U \omega + \rho_V \omega = \omega.$$

This shows that j^* is surjective, and the Mayer-Vietoris sequence is a short exact sequence. ■

The maps i^*, j^* induce linear maps $i^\# : H^k(X) \to H^k(U) \bigoplus H^k(V)$ and $j^\# : H^k(U) \bigoplus H^k(V) \to H^k(U \cap V)$ in the de Rham cohomology, defined by

$$i^\#[\sigma] = [i^*\sigma] \quad \text{and} \quad j^\#([\omega], [\tau]) = ([j^*\omega], [j^*\tau]).$$

To check that they are well-defined, we need to show that i^*, j^* send closed forms to closed forms, exact forms to exact forms. This follows from the commutativity of d with the pullback.

Next, we define a map $d^\# : H^k(U \cap V) \to H^{k+1}(X)$, called the *connecting homomorphism*, as follows:

$$
\begin{tikzcd}
\Omega^{k+1}(X) \arrow[r, hook, "i^*"] & \Omega^{k+1}(U)\bigoplus\Omega^{k+1}(V) & \\
& \Omega^{k}(U)\bigoplus\Omega^{k}(V) \arrow[u, "d"] \arrow[r, two heads, "j^*"] & \Omega^{k}(U\cap V),
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha \arrow[r, hook, "i^*"] & (d\zeta_U, d\zeta_V) \arrow[r, mapsto, two heads, "j^*"] & 0 \\
& (\zeta_U,\zeta_V) \arrow[u, mapsto, "d"] \arrow[r, mapsto, two heads, "j^*"] & \zeta. \arrow[u, mapsto, "d"]
\end{tikzcd}
$$

(1) We start with a closed form $\zeta\in\Omega^k(U\cap V)$. Let $\{\rho_U,\rho_V\}$ be a partition of unity subordinate to $\{U,V\}$. We can extend $\rho_U\zeta$ by zero from $U\cap V$ to a k-form ζ_V on V, and extend $\rho_V\zeta$ by zero from $U\cap V$ to a k-form ζ_U on U. Then

$$j^*(-\zeta_U,\zeta_V)=\zeta_V|_{U\cap V}+\zeta_U|_{U\cap V}=(\rho_U+\rho_V)\zeta=\zeta.$$

(2) The exterior derivative sends $(-\zeta_U,\zeta_V)$ to $(-d\zeta_U,d\zeta_V)\in\Omega^{k+1}(U)\bigoplus\Omega^{k+1}(V)$. Observe that $(-d\zeta_U,d\zeta_V)$ lies in $\ker j^*$ due to the commutativity of j^* with d. Specifically,

$$j^*(-d\zeta_U,d\zeta_V)=j^*d(-\zeta_U,\zeta_V)=dj^*(-\zeta_U,\zeta_V)=d\zeta=0.$$

(3) Since $\ker j^*=\operatorname{im} i^*$, $(-d\zeta_U,d\zeta_V)=i^*\alpha$ for some $\alpha\in\Omega^{k+1}(X)$. The choice of α is unique because i^* is injective. Note that α is a closed form since i^* is injective and

$$i^*d\alpha=di^*\alpha=d(-d\zeta_U,d\zeta_V)=(0,0).$$

We define the connecting homomorphism to be $d^\#[\zeta]=[\alpha]$.

The definition of the connecting homomorphism relies on two choices: the choice of ζ to represent the cohomology class $[\zeta]$, and the choice of the partition of unity (ρ_U,ρ_V). To show that $d^\#$ is well-defined, one must show that $[\alpha]$ does not depend on these choices. This is requires a bit of diagram chasing. The readers are encouraged to try this out themselves.

By the discussion above, we obtain a sequence in the de Rham cohomlogy, also called the *Mayer-Vietoris sequence*:

$$
\begin{tikzcd}
H^{k+1}(X) \arrow[r, "i^\#"] & \cdots & \\
H^k(X) \arrow[r, "i^\#"] & H^k(U)\oplus H^k(V) \arrow[r, "j^\#"] & H^k(U\cap V) \arrow[ull, rounded corners, "d^\#"] \\
& \cdots \arrow[r, "j^\#"] & H^{k-1}(U\cap V) \arrow[ull, rounded corners, "d^\#"]
\end{tikzcd}
$$

Proposition 3.9. *The Mayer-Vietoris sequence is exact.*

Proof. The proof is a lot of diagram-chasing, which is not very enlightening. To give a sense of how this works, however, we will show exactness at $H^k(U\cap V)$.

First, we show that $\operatorname{im} j^\#\subset\ker d^\#$. Let $j^\#([\omega],[\tau])$ be an element in $\operatorname{im} j^\#$. By the definition of $j^\#$, we have $d^\# j^\#([\omega],[\tau])=d^\#[j^*(\omega,\tau)]$. Choose (ω,τ) to be the element which j^* maps to $j^*(\omega,\tau)$. Note that ω and τ are closed forms, so the exterior derivative sends them to 0. Since i^* is injective, we must have $d^\#[j^*(\omega,\tau)]=[0]$, and thus $j^\#([\omega],[\tau])\in\ker d^\#$. This can be made clear by the following diagram:

$$
\begin{tikzcd}
0 \arrow[r, hook, "i^*"] & (d\omega,d\tau)=0 & \\
& (\omega,\tau) \arrow[u, mapsto, "d"] \arrow[r, mapsto, two heads, "j^*"] & j^*(\omega,\tau).
\end{tikzcd}
$$

Next, we show that $\ker d^{\#} \subset \operatorname{im} j^{\#}$. Suppose $d^{\#}[\zeta] = [\alpha] = 0$. This means that there exists $\alpha' \in H^{k-1}(X)$ such that $d\alpha' = \alpha$. The calculation of $d^{\#}[\zeta]$ can be traced with the following diagram:

$$\begin{tikzcd}
\alpha \arrow[r, hook, "i^*"] & (d\zeta_U, d\zeta_V) & \\
\alpha' \arrow[u, "d"] & (\zeta_U, \zeta_V) \arrow[u, "d"] \arrow[r, maps to, two heads, "j^*"] & \zeta.
\end{tikzcd}$$

We claim that $(\zeta_U, \zeta_V) - i^*\alpha'$ is a closed form in $\Omega^k(U) \bigoplus \Omega^k(V)$ that maps to ζ under j^*, because

$$\begin{aligned}
&d\big((\zeta_U, \zeta_V) - i^*\alpha'\big) = d(\zeta_U, \zeta_V) - di^*\alpha' = d(\zeta_U, \zeta_V) - i^*d\alpha' = d(\zeta_U, \zeta_V) - i^*\alpha = 0, \\
&j^*\big((\zeta_U, \zeta_V) - i^*\alpha'\big) = j^*(\zeta_U, \zeta_V) - j^*i^*\alpha' = \zeta.
\end{aligned}$$

Thus, $j^{\#}[(\zeta_U, \zeta_V) - i^*\alpha'] = [\zeta]$, so $[\zeta] \in \operatorname{im} j^{\#}$. ■

The Mayer-Vietoris sequence allows us to treat $H^k(X)$ as a "function" of $H^k(U)$, $H^k(U)$ and $H^k(U \cap V)$. It is especially useful when some of the terms in the long exact sequence is 0, in which case exactness implies that the neighboring maps must be injections, surjections, or even isomorphisms. In Section 4.1 we will see the power of Mayer-Vietoris in computing the cohomology of the sphere S^n.

3.3 Homotopy Invariance

One of the reasons that we care about de Rham cohomology is that it allows us to differentiate between different spaces under homotopy. Homotopy is usually defined with continuous maps, but since we are mainly occupied with smoothness in the discussion of differential forms, we will define homotopy with smooth maps. Section 3.3.1 shows that we can freely pass between smooth homotopy and continuous homotopy. Section 3.3.2 builds up to the homotopy invariance of de Rham cohomology.

3.3.1 Smooth Homotopy

Definition 3.10 (smooth homotopy)**.** Let $U, V \subseteq \mathbb{R}^n$ be vector spaces. Two C^∞ maps $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ are *smoothly homotopic* if there exists a C^∞ map $F : U \times \mathbb{R} \to V$ such that for all $x \in U$,

$$F(x, 0) = f(x), \text{ and } F(x, 1) = g(x).$$

If f and g are smoothly homotopic, we write $f \sim g$.

Definition 3.11 (smooth homotopy equivalence)**.** A C^∞ map $f : U \to V$ is a *smooth homotopy equivalence* if there exists a C^∞ map $g : V \to U$, such that

$$f \circ g \sim \mathbb{1}_V \text{ and } g \circ f \sim \mathbb{1}_U.$$

In this case, we say that U and V have the same *homotopy type*.

Smooth homotopy is a stronger condition than the continuous homotopy that we are familiar with: all smooth functions are continuous, but not all continuous functions are smooth. However, we can always approximate a continuous function by a smooth function, up to homotopy. The following proposition allows us to pass from continuous maps and homotopies to smooth maps and smooth homotopies. For an exposition of the proof, see [4].

Proposition 3.12 (Smoothening Lemma)**.**
(i) If $f : U \to V$ is a continuous map between Euclidean spaces, then there is a smooth map $g : U \to V$ such that f and g are homotopic.
(ii) If g, g' are smooth and homotopic, then they are smoothly homotopic.

Corollary 3.13. *If U and V are homotopic equivalent, then they are smoothly homotopic equivalent.*

Proof. Since U and V are homotopic equivalent, there exists continuous maps $f : U \to V$ and $g : V \to U$ such that

$$f \circ g \sim \mathbb{1}_V \ \text{ and } \ g \circ f \sim \mathbb{1}_U.$$

By Proposition 3.12 part (i), we can find C^∞ maps $f' : U \to V$ and $g' : V \to U$ such that f' is homotopic to f and g' is homotopic to g. Thus,

$$f' \circ g' \sim f \circ g \sim \mathbb{1}_V \ \text{ and } \ g' \circ f' \sim g \circ f \sim \mathbb{1}_U.$$

By Proposition 3.12 part (ii), $f' \circ g'$ and $g' \circ f'$ are smoothly homotopic to $\mathbb{1}_V$ and $\mathbb{1}_U$, respectively. By definition, U and V are smoothly homotopic. ■

3.3.2 Homotopy Invariance

Let U, V be open sets in $\mathbb{R}^n$. For a C^∞ map $F : V \to U$, its pullback $F^* : \Omega^k(U) \to \Omega^k(V)$ induces a linear map $F^\# : H^k(M) \to H^k(N)$, defined by

$$F^\#[\omega] = [F^*(\omega)].$$

To see why $F^\#$ is well-defined, we check that $F^\#$ sends closed forms to closed forms, exact forms to exact forms. With this definition, Proposition 2.26 extends to

Lemma 3.14. *Let $f : U \to V$ and $g : V \to W$ be smooth maps on real vector spaces, then*

$$(g \circ f)^\# = f^\# \circ g^\#.$$

Proposition 3.15. *If $f, g : \ U \to V$ are (smoothly) homotopic smooth maps, then the induced cohomology maps $f^\#, g^\# : \ H^k(V) \to H^k(U)$ are equal.*

Proof. By the definition of smooth homotopy, there exists a smooth map $F : U \times \mathbb{R} \to V$, such that

$$F(x, 0) = f(x), \ \ F(x, 1) = g(x), \quad \forall x \in U. \tag{3}$$

Define $i_t : U \to U \times \mathbb{R}$ with $i_t(x) = (x, t)$. Then (3) becomes $F \circ i_0 = f$ and $F \circ i_1 = g$. By Lemma 3.14,

$$f^\# = i_0^\# \circ F^\#, \ \ g^\# = i_1^\# \circ F^\#.$$

To prove $f^\#$ and $g^\#$ are the same, it amounts to showing that $i_0^\#$ and $i_1^\#$ are the same, i.e., for any closed k-form ω on $U \times \mathbb{R}$, $i_0^*(\omega)$ and $i_1^*(\omega)$ differ by an exact form. We can write

$$\omega = dt \wedge \psi_t + \omega_t,$$

where ψ_t is a time-dependent $(k-1)$-form on U, and ω_t is a time-dependent k-form on U. Since ω is closed,

$$0 = d\omega = d(dt \wedge \psi_t + \omega_t) = -dt \wedge d\psi_t + d\omega_t = dt \wedge (-d\psi_t + \frac{d\omega_t}{dt}).$$

It follows that $d\psi_t = \omega_t'$, where ω_t' denotes the time derivative of ω_t. Now observe that

$$d\int_0^1 \psi_t = \int_0^1 d\psi_t = \int_0^1 \omega_t' = \omega_1 - \omega_0 = i_1^*(\omega) - i_0^*(\omega).$$

Thus, the induced maps $f^\#$ and $g^\#$ are the same. ■

Theorem 3.16 (Homotopy Invariance of de Rham Cohomology)**.** *If U and V are homotopy equivalent, then $H^k(V) \cong H^k(U)$ for each k.*

Proof. By Corollary 3.13, U and V are smoothly homotopy equivalent, so there exists smooth maps $f : U \to V$ and $g : V \to U$, such that

$$f \circ g \sim \mathbb{1}_V \text{ and } g \circ f \sim \mathbb{1}_U.$$

We can then induce linear maps $f^\# : H^k(U) \to H^k(V)$, $g^\# : H^k(V) \to H^k(U)$. By Proposition 3.15,

$$f^\# \circ g^\# = \mathbb{1}_{H^k(V)} \text{ and } g^\# \circ f^\# = \mathbb{1}_{H^k(U)}.$$

It follows that $H^k(V) \cong H^k(U)$. ■

Corollary 3.17 (Homeomorphim Invariance of de Rham Cohomology)**.** *If U and V are homeomorphic, then $H^k(U) \cong H^k(V)$ for each k.*

Corollary 3.18 (Poincaré Lemma)**.** *Let U be a star-shaped open set in $\mathbb{R}^n$. Since U is contractible,*

$$H^k(U) = \begin{cases} \mathbb{R} & \text{if } k = 1, \\ 0 & \text{if } k \geq 1. \end{cases}$$

4 Applications

4.1 Cohomology of the Sphere

In this section, we will use the Mayer-Vietoris sequence to deduce the de Rham cohomology of the sphere S^n, for any n. Let's start when $n = 1$. From Proposition 3.2 and Proposition 3.3, we know that $H^k(S^1) = \mathbb{R}$ if $k = 0$ and $H^k(S^1) = 0$ if $k \geq 2$. Our task, therefore, is to find $H^1(S^1)$.

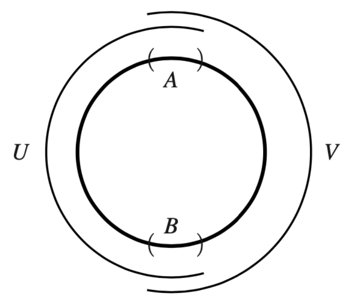

Figure 5: Covering the circle with two open arcs.

Cover the circle with two open arcs U, V as shown in Figure 5. Their intersections are two disjoint open arcs, which we call A and B. By homotopy invariance, the de Rham cohomlogy of U and V are isomorphic to that of $\mathbb{R}$. Moreover, $U \cap V = A \sqcup B$, so the de Rham cohomology of $U \cap V$ is isomorphic to that of $\mathbb{R} \sqcup \mathbb{R}$. Thus, we obtain a Mayer-Vietoris sequence

$$0 \to \mathbb{R} \xrightarrow{i^{\#}} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j^{\#}} \mathbb{R} \oplus \mathbb{R} \xrightarrow{d^{\#}} H^1(S^1) \to 0.$$

Because $i^{\#}$ is injective, $\dim(\operatorname{im} i^{\#}) = \dim(\ker j^{\#}) = 1$. This in turn means that $\dim(\operatorname{im} j^{\#}) = \dim(\ker d^{\#}) = 1$. Hence, $\dim(\operatorname{im} d^{\#}) = 1$. Since $d^{\#}$ is surjective, $\dim(H^1(S^1)) = \dim(\operatorname{im} d^{\#}) = 1$, which implies that $H^1(S^1) = \mathbb{R}$. Therefore, the cohomology of the circle is given by

$$H^k(S^1) = \begin{cases} \mathbb{R} & \text{if } k = 0, 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

Next, we consider the de Rham cohomology of the sphere S^2. We can cover S^2 by 2 "domes" U and V, which are homeomorphic to the disk D^1. The intersection of U and V is homeomorphic to $S^1 \times (0, 1)$. By homotopy invariance, the cohomology of U and V are isomorphic to that of $\mathbb{R}^2$, the cohomology of $U \cap V$ to that of S^1. This fits into the Mayer-Vietoris sequence

$$0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R} \to H^1(S^2) \to 0 \to \mathbb{R} \to H^2(S^2) \to 0.$$

Using the same dimension argument as above, we conclude that $H^1(S^2) = 0$ and $H^2(S^2) = \mathbb{R}$.

For higher dimensions, we can proceed in the same way via induction. This gives us

Proposition 4.1. *The de Rham cohomology of an n-sphere is*

$$H^k(S^n) = \begin{cases} \mathbb{R} & \textit{if } k = 1, n, \\ 0 & \textit{otherwise.} \end{cases}$$

Corollary 4.2. *S^n and S^m are homotopic equivalent if and only if $n = m$.*

Corollary 4.3. *Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.*

Proof. Suppose $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic. Then $\mathbb{R}^n - \{0\}$ and $\mathbb{R}^m - \{0\}$ are homeomorphic. Since $\mathbb{R}^k - \{0\} \cong S^k$ for all k, this implies that S^n and S^m are homeomorphic, and therefore $n = m$ by Corollary 4.2. The other direction is trivial. ■

4.2 Brower's Fixed Point Theorem

Proposition 4.4. *Let $f : D^n \to D^n$ be a continuous map, where D^n denotes the n-disk. Then f has a fixed point, i.e. there exists $x \in D^n$ such that $f(x) = x$.*

Proof. For the sake of contradiction, assume that f doesn't have a fixed point, that is, $f(x)$ and x are distinct for all $x \in D^n$. Then there is a unique line segment from $f(x)$ to x, which intersects the boundary twice (see Figure 6).

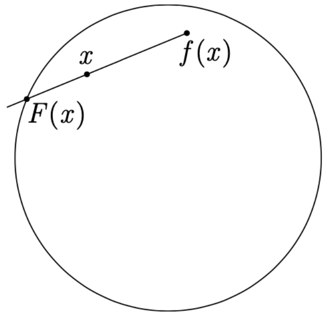

Figure 6: Construction of $F(x)$

Denote the intersection closer to x by $F(x)$. Explicitly, we have $F(x) = x + ut$, where $u = \frac{x - f(x)}{||x - f(x)||}$, and

$$t = -x \cdot u + \sqrt{1 - ||x|^2 + (x \cdot u)^2}.$$

The function F is continuous with $F|_{S^{n-1}} = \mathrm{Id}|_{\mathrm{S}^{n-1}}$. Using F, we will construct a homotopy from the identity function on $\mathbb{R}^n - \{0\}$ to a constant function as follows. Let $r : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ be the function that normalizes its argument, that is, $r(x) = \frac{x}{||x||}$. Then r is homotopic to the identity map on $\mathbb{R}^n - \{0\}$. Define a homotopy

$$\begin{aligned} H : \mathbb{R}^n - \{0\} \times [0,1] &\to \mathbb{R}^n - \{0\} \\ (x,t) &\mapsto F\big((1-t)r(x)\big), \end{aligned}$$

where $H(x, 0) = r(x)$ and $H(x, 1) = F(0)$. H is a composition of continuous functions, and therefore continuous. Hence, $\mathbb{R}^n - \{0\}$ is contractible. By Corollary 3.18, $H^{n-1}(\mathbb{R}^n - \{0\}) = 0$, which contradicts Proposition 4.1. ■

4.3 Hopf Invariant

Using the de Rham cohomology, we can construct another homotopy invariant, called the Hopf invariant. The mechanism involved goes beyond the scope of this paper. We therefore sacrifice rigor, in attempt to give a more intuitive understanding of the Hopf invariant.

In vector calculus, we learned about Stoke's thoerem for $\mathbb{R}^2$ and $\mathbb{R}^3$. It states that the integral of the curl of a vector field over some surface is equal to the line integral of the vector field around the boundary of the surface. This can be generalized to $\mathbb{R}^n$ (and, in fact, to any orientated manifold), as follows.

Proposition 4.5 (Stoke's Theorem)**.** *Let U be a subspace of $\mathbb{R}^n$, and ω be any smooth $(n-1)$-form on U. Then*

$$\int_U d\omega = \int_{\partial U} \omega.$$

Let $f : S^{2n-1} \to S^n$ be a C^∞ map, where $n \geq 1$. By Proposition 4.1, $H^n(S^n) = \mathbb{R}$ and therefore has a generator α. The pullback $f^*\alpha$ is an n-form on S^{2n-1}. Since $H^n(S^{2n-1}) = 0$, every closed n-form on S^{2n-1} is exact. Let ω be the $(n-1)$-form on S^{2n-1} such that $f^*\alpha = d\omega$.

Definition 4.6 (Hopf invariant)**.** The *Hopf invariant of f* is defined to be

$$H(f) = \int_{S^{2n-1}} \omega \wedge d\omega.$$

First, we need to check that the Hopf invariant is well-defined. Let ω' be another $(n-1)$-form on S^{n-1} such that $f^*\alpha = d\omega'$. Note that $d(\omega - \omega') = d\omega - d\omega' = 0$. Then

$$\begin{aligned}\int_{S^{2n-1}} \omega \wedge d\omega - \int_{S^{2n-1}} \omega' \wedge d\omega' &= \int_{S^{2n-1}} (\omega - \omega') \wedge d\omega \\ &= \int_{S^{2n-1}} d((\omega - \omega') \wedge \omega) \\ &= 0.\end{aligned}$$

The last equality follows from Stoke's theorem.

Remark 4.7. When n is odd, we have an even-dimensional ω, which means that

$$\omega \wedge d\omega = \frac{1}{2}(d\omega \wedge \omega + \omega \wedge d\omega) = \frac{1}{2}d(\omega \wedge \omega).$$

By Stoke's theorem, $H(f) = \int_{S^{2n-1}} \omega \wedge d\omega = 0$.

Proposition 4.8. *If two maps $f, g : S^{2n-1} \to S^n$ are homotopic, then they have the same Hopf invariant.*

Proof. By Remark 4.7, we may assume that n is even. There exists a homotopy $F : S^{2n-1} \times [0, 1] \to S^n$ from f to g. Define $i_t : S^{2n-1} \to S^{2n-1} \times \mathbb{R}$, where $i_t(x) = (x, t)$. We have

$$F \circ i_0 = f_0 \quad \text{and} \quad F \circ i_1 = g.$$

Let α be a generator of $H^n(S^n)$. Then $F^*\alpha = d\omega$ for some $(n-1)$-form ω on $S^{2n-1} \times I$. Let $i_0^*\omega = \omega_0$ and $i_1^*\omega = \omega_1$. Then we have

$$f_0^*\alpha = i_0^* \circ F^*\alpha = i_0^* \circ d\omega = di_0^*\omega = d\omega_0.$$

Similarly, $f_1^*\alpha = d\omega_1$. Observe that

$$\omega_0 \wedge d\omega_0 = i_0^*\omega \wedge di_0^*\omega = i_0^*(\omega \wedge d\omega).$$

Using this fact and Stoke's theorem, we get

$$\begin{aligned}H(f) - H(g) &= \int_{S^{2n-1}} \omega_1 \wedge d\omega_1 - \int_{S^{2n-1}} \omega_0 \wedge d\omega_0 \\ &= \int_{S^{2n-1}} i_1^*(\omega \wedge d\omega) - \int_{S^{2n-1}} i_0^*(\omega \wedge d\omega) \\ &= \int_{S^{2n-1}\times\{1\}} \omega \wedge d\omega - \int_{S^{2n-1}\times\{0\}} \omega \wedge d\omega \\ &= \int_{\partial(S^{2n-1}\times[0,1])} \omega \wedge d\omega \\ &= \int_{S^{2n-1}\times[0,1]} d\omega \wedge d\omega \\ &= \int_{S^{2n-1}\times[0,1]} F^*(\alpha \wedge \alpha) \\ &= 0.\end{aligned}$$

Therefore, f and g has the same Hopf invariant. ■

Example 4.9 (Hopf invariant of the Hopf fibration)**.** When $n = 2$, we have the Hopf map (also known as the Hopf fibration) $\eta : S^3 \to S^2$. Here, S^3 is the unit sphere in $\mathbb{R}^4$, which we identify with $\mathbb{C}^2$. Furthermore, we identify S^2 with the complex projective line $\mathbb{CP}^1$ via the stereographic projection. Define η as the map

$$\eta(z_0, z_1) = [z_0 : z_1], \quad \text{for } z_0, z_1 \in \mathbb{C}.$$

Our task is to compute the Hopf invariant for η.

To start off, we want to find a generator α of $H^2(\mathbb{CP}^1)$. This is not so easy: we first have to find a generator of $H^2(S^2)$ and then project it onto $\mathbb{CP}^1$. We will put a black box around this (for interested readers, refer to [1]). Now, assume that we have found a generator α in $H^2(\mathbb{CP})$, with the form

$$\alpha = \frac{i}{2\pi} \frac{(z_1 dz_0 - z_0 dz_1)(\bar{z_1} d\bar{z_0} - \bar{z_0} d\bar{z_1})}{(|z_0|^2 + |z_1|^2)}.$$

Our next step is to find a 1-form ω on S^3 such that $\eta^*\alpha = d\omega$. Let $z_0 = x_1 + ix_2$ and $z_1 = x_3 + ix_4$ be the coordinates on $\mathbb{C}^2$. Using the definition of a pullback, we compute that

$$\eta^*\alpha = \frac{1}{\pi}(dx_1 dx_2 + dx_3 dx_4) = \frac{1}{\pi} d(x_1 dx_2 + x_3 dx_4)$$

So we can take $\omega = \frac{1}{\pi}(x_1 dx_2 + x_3 dx_4)$. Therefore, the Hopf invariant of η is

$$\begin{aligned}
H(f) &= \int_{S^3} \omega \wedge d\omega \\
&= \frac{1}{\pi^2} \int_{S^3} (x_1 dx_2 + x_3 dx_4) \wedge (dx_1 dx_2 + dx_3 dx_4) \\
&= \frac{2}{\pi^2} \int_{S^3} x_1 dx_2 dx_3 dx_4 + x_3 dx_1 dx_2 dx_4 \\
&= \frac{1}{\pi^2} \int_{S^3} x_1 dx_2 dx_3 dx_4.
\end{aligned}$$

The last equality follows by symmetry. To calculate this integral, we can use spherical coordinates

$$\begin{aligned}
x_1 &= \sin\zeta \sin\psi \cos\theta, \\
x_2 &= \sin\zeta \sin\psi \sin\theta, \\
x_3 &= \sin\zeta \cos\psi, \\
x_4 &= \cos\psi.
\end{aligned}$$

where $0 \le \psi \le \pi$, $0 \le \psi \le \pi$, and $0 \le \theta \le 2\pi$. The Hopf invariant becomes

$$\frac{2}{\pi^2} \int_{S^3} x_1 dx_2 dx_3 dx_4 = \frac{2}{\pi^2} \int_0^\pi \int_0^\pi \int_0^{2\pi} \sin^4\zeta \sin^3\psi \cos^2\theta d\theta d\psi d\zeta = \frac{2}{\pi^2} \cdot \frac{3\pi}{4} \cdot \frac{4}{3} \cdot \frac{\pi}{2} = 1.$$

By Proposition 4.8, the Hopf map η is not null-homotopic. Somehow, we can "wrap" a 3-sphere around a 2-sphere. For an interesting visualization, see [2].

References

[1] Bott, Raoul, and Loring W. Tu. *Differential Forms in Algebraic Topology*. New York: Springer, 2011.
[2] Johnson, Niles. "Hopf Fibration – Fibers and Base." YouTube, August 22, 2011. Accessed December 18, 2021. https://www.youtube.com/watch?v=AKotMPGFJYk.
[3] Lee, John Marshall. *Introduction to Smooth Manifolds*. Springer, 2013.
[4] Madsen, I. H., and Jørgen Tornehave. *From Calculus to Cohomology: De Rham Cohomology and Characteristic Classes*. Cambridge: Cambridge University Press, 2015.
[5] Pankka, Pekka. "Introduction to De Rham Cohomology." Accessed December 18, 2021. https://www.mv.helsinki.fi/home/pankka/deRham2013.
[6] Tu, Loring W. *An Introduction to Manifolds*. New York: Springer, 2011.