Metrizable spaces

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Intended audience: students who know the material in MATH GU4051 Topology until (including) homework #3, which covers compact spaces and Hausdorff spaces.

Contents

1 Ruling out metrizability **2**

1.1 "Separation" properties . . . 2

1.2 "Complexity" properties . . . 4

2 Sufficient conditions for metrizability **7**

2.1 Urysohn's metrization theorem statement . . . 7

2.2 Topological manifolds are metrizable . . . 8

2.3 Proving Urysohn's metrization theorem . . . 9

3 Equivalent characterization of metrizability **16**

3.1 Urysohn metrization theorem fails . . . 16

3.2 Nagata-Smirnov metrization theorem . . . 17

The standard examples to build intuitions when learning point-set topology are metric spaces. But as we will soon see, not all topological spaces are topologically equivalent to metric spaces, or more formally, metrizable.

Definition 1 (Metrizable topological space)
A topological space $(X, \mathcal{T})$ *is* ***metrizable*** *if* $\mathcal{T}$ *is induced by some metric.*

From this definition, there is a simple way to show that a topological space is metrizable: give a metric that induces its topology. Similarly, to show that a topological space is not metrizable, it suffices to derive a contradiction from the assumption that some metric induces its topology.

However, working with metrics to show that a topological space is metrizable or not is cumbersome. When working with topological spaces, we are not aware of metrics inducing topologies unless spaces are clearly metrizable in the first place. Nor are we generally aware of contradictions arising from assuming there are metrics inducing certain topologies.

When working with topological spaces, we are aware of some of their "intrinsic" topological properties such as being compact or Hausdorff. Therefore, it would be great if we could tell whether a topological space is metrizable or not just by looking at some of its "intrinsic" properties. This paper presents results that allow us to do this.

We start with "intrinsic" topological properties that allow us to rule out metrizability. Next, we talk about "intrinsic" topological properties that suffice for metrizability. Finally, we will talk about "intrinsic" topological properties that are equivalent to metrizability. Let's start!

1 Ruling out metrizability

1.1 "Separation" properties

The first "intrinsic" topological properties that allow us to rule out metrizability are "separation" properties. The first such "separation" property is having separated points, or, more formally, being Hausdorff. Any metrizable space is Hausdorff. This allows us to rule out metrizability for non-Hausdorff spaces.

Definition 2 (Hausdorff topological space)
*A topological space $(X, \mathcal{T})$ is **Hausdorff** if for any two distinct points x, y in X there exist disjoint open sets $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$.*

Lemma 1 (Metrizable $\Rightarrow$ Hausdorff)
Any metrizable topological space $(X, \mathcal{T})$ is Hausdorff.

Proof.

Let $(X, \mathcal{T})$ be a metrizable topological space. Let x, y be distinct points in X and let d be a metric inducing $\mathcal{T}$. Let $\varepsilon := \frac{d(x,y)}{2}$. Since $x \neq y$, $d(x, y) > 0$ and we can talk about the open balls $B^d_\varepsilon(x)$ and $B^d_\varepsilon(y)$.

For any $a \in X$ $d(a, a) = 0$, so $x \in B^d_\varepsilon(x)$ and $y \in B^d_\varepsilon(y)$. And since $d(x, y) \not< \varepsilon$, $y \notin B^d_\varepsilon(x)$ and $x \notin B^d_\varepsilon(y)$. And by the triangle inequality of the metric d, $B^d_\varepsilon(x)$ and $B^d_\varepsilon(y)$ are disjoint. Finally, since d induces the topology $\mathcal{T}$, $B^d_\varepsilon(x)$ and $B^d_\varepsilon(y)$ are basic open sets in $\mathcal{T}$ and so open.

So x, y are separated by open sets. Since x, y were arbitrary, this shows that $(X, \mathcal{T})$ is Hausdorff. $\square$

As an example, we can apply our result to rule out metrizability for the Siepiński space.

Example 1 (Sierpiński space is not metrizable)
The Sierpiński space is not Hausdorff and so not metrizable.

Proof.

The Sierpiński space has underlying set $\{0, 1\}$ and topology $\{\varnothing, \{1\}, \{0, 1\}\}$. By inspection, there is no open set U in the topology such that $0 \in U$ and $1 \notin U$. So in particular there are no disjoint open sets U, V such that $0 \in U$ and $1 \in V$. $\square$

And we can also apply our result to rule out metrizability for $\mathbb{R}$ with the countable complement topology.

Example 2 ($\mathbb{R}_{cc}$ is not metrizable)
$\mathbb{R}$ with the countable complement topology is not Hausdorff and so not metrizable.

Proof.

Let U, V be open sets. Then $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus V$ are countable sets. So $(\mathbb{R} \setminus U) \cup (\mathbb{R} \setminus V) = \mathbb{R} \setminus (U \cap V)$ is countable. Since $\mathbb{R}$ is uncountable, this implies that $U \cap V$ is uncountable and nonempty. Thus any two open sets in R_{cc} intersect. In particular, $\mathbb{R}_{cc}$ is not Hausdorff. $\square$

Besides Hausdorffness, there is another “separation” property that allows us to rule out metrizability. It is the “separation” property of having separated points and closed sets, or, more formally, being regular. Any metrizable space is regular, what allows us to rule out metrizability for non-regular spaces.

Definition 3 (Regular topological space)
A topological space $(X, \mathcal{T})$ is ***regular*** *if for any closed set C and any point $x \notin C$, there exist disjoint open sets $U, V \in \mathcal{T}$ such that $C \subseteq U$ and $x \in V$.*

Lemma 2 (Metrizable $\Rightarrow$ Regular)
Any metrizable topological space $(X, \mathcal{T})$ is regular.

Proof.

Let C be a closed set and let $x \notin C$. Since C is closed, $C = \overline{C}$. By the limit-point characterization of closure, there is an open set U with $x \in U$ such that $U \cap C = \varnothing$.

Since $(X, \mathcal{T})$ is metrizable, there is some metric d such that U is the union of open balls with the metric d. One of these open balls contains x, so we have $B_r(x) \cap C = \varnothing$ for some radius $r > 0$. So $d(x, c) \geq r$ for any $c \in C$. So the set $\{d(x, c) \mid c \in C\}$ has a greatest lower bound $\varepsilon \geq r > 0$.

Consider $\bigcup\limits_{c \in C} B_\varepsilon(c)$ and $B_\varepsilon(x)$. Clearly, $C \subseteq \bigcup\limits_{c \in C} B_\varepsilon(c)$ and $x \in B_\varepsilon(x)$. Moreover, by the triangle inequality of the metric d, these two sets are disjoint. Finally, these two sets are open because d induces $\mathcal{T}$.

So C and x are separated by open sets. Since C, x and $(X, \mathcal{T})$ were arbitrary, this means that any metrizable space is regular. $\square$

We cannot use non-Hausdorffness to rule out metrizability for $\mathbb{R}_{ccs}$ since it is Hausdorff. But our last result allows us to rule out metrizability for this space nevertheless.

Example 3 ($\mathbb{R}_{ccs}$ is not metrizable)
$\mathbb{R}$ with open sets $U \setminus C$ where U is open in the standard topology and C is a countable subset of U is Hausdorff, but not regular, and so not metrizable.

Proof.

First, let's show that $\mathbb{R}_{ccs}$ is Hausdorff. Let x, y be distinct points in $\mathbb{R}$. Since $\mathbb{R}_{std}$ is trivially metrizable, it is Hausdorff. So for any two distinct points $x, y \in \mathbb{R}$ there are disjoint open sets U, V in $\mathbb{R}_{std}$ such that $x \in U$ and $y \in V$. Note that any open set in $\mathbb{R}_{std}$ is open in $\mathbb{R}_{ccs}$ taking $C = \varnothing$. So U, V are also disjoint open sets in $\mathbb{R}_{ccs}$ such that $x \in U$ and $y \in V$.

Now, let's show that $\mathbb{R}_{ccs}$ is not regular. Consider the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$. Since $\mathbb{Q}$ is a countable set, this is an open set in $\mathbb{R}_{ccs}$. So $\mathbb{Q}$ is a closed set in $\mathbb{R}_{ccs}$. Let t be an irrational number. Since the rationals are dense, there is some rational number in any open set containing t. So in particular, there is no pair of disjoint open sets U, V such that $\mathbb{Q} \subseteq U$ and $t \in V$. $\square$

1.2 “Complexity” properties

In addition to the “separation” properties discussed above, there are “complexity” properties that help us rule out metrizability. Both “local complexity” and “global complexity” properties come into play. First, let’s talk about the “local complexity” property of first-countability.

A first-countable space is one in which openness around a point can be described with countably many open sets. Most importantly for our purposes, first-countability is necessary for metrizability. So, whenever a space is not first-countable, we can rule metrizability out.

Definition 4 (Neighborhood basis, first-countable topological space)
Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. A ***neighborhood basis of*** *x is a collection $\mathcal{B}_x = \{U_i\}_{i \in I}$ of open sets in $\mathcal{T}$ such that $x \in U_i$ for all $i \in I$ and such that for any open set U with $x \in U$, there is some $U_i \in \mathcal{B}_x$ with $U_i \subseteq U$. $(X, \mathcal{T})$ is* ***first-countable*** *if there is a countable neighborhood basis of each point in X.*

Lemma 3 (Metrizable $\Rightarrow$ First-countable)
Any metrizable topological space is first-countable.

Proof.

Let $(X, \mathcal{T})$ be a metrizable topological space and let $x \in X$. Since the space is metrizable, there is a metric d that induces $\mathcal{T}$.

Let’s show that

$$\mathcal{B}_x = \left\{ B_{\frac{1}{n}}(x) \mid n \in \mathbb{N} \right\}$$

is a countable neighborhood basis of x.

Note first that $\mathcal{B}_x$ is countable because the natural numbers are countable. Moreover, x is clearly a member of each set in $\mathcal{B}_x$. And since d induces $\mathcal{T}$, the sets in $\mathcal{B}_x$ are open.

Since d induces $\mathcal{T}$, any open set U with $x \in U$ is a union of open balls with the metric d. One of these open balls contains x, so we have $B_r(x) \subseteq U$ for some radius r. Pick some number m such that $mr > 1$. Then $B_{\frac{1}{m}}(x) \subseteq B_r(x) \subseteq U$.

$\square$

We can apply our new result to rule out metrizability for the uncountable product of the unit interval. Note again that our new result allows us to rule out metrizability for new spaces. $[0, 1]^{[0,1]}$ is Hausdorff and regular, so our previous results do not apply to it.

Example 4 (Uncountable product of the unit interval is not metrizable)
$[0, 1]^{[0,1]}$ *is Hausdorff and regular, but not first-countable and so not metrizable.*

Proof.

Hausdorffness and regularity are closed under the subspace operation on spaces. So $[0, 1]$ as a subspace of $\mathbb{R}$ is Hausdorff and regular. Furthermore, Hausdorffness and regularity are closed under arbitrary products. So, $[0, 1]^{[0,1]}$ is Hausdorff and regular.

However, $[0,1]^{[0,1]}$ is not first-countable because 0 does not have a countable neighborhood basis. Suppose towards contradiction that 0 does have a countable neighborhood basis $\{U_i\}_{i\in I}$. By definition of product topology, $U_i = \prod_{x\in[0,1]} U_{i,x}$, where finitely many $U_{i,x}$ are different from $[0,1]$.

Consider

$$\{x \in [0,1] \mid U_{i,x} \neq [0,1] \text{ for some } i \in I\}$$

It is the countable union of countable union of finite sets, $\{x \in [0,1] \mid U_{i,x} \neq [0,1]\}$ for each $i \in I$. So it is countable. So it is a proper subset of $[0,1]$. Pick $x_0 \in [0,1]$ outside it.

Let $H := \prod_{x\in[0,1]} H_x$, where $H_{x_0} = [0,1/2)$ and $H_x = [0,1]$ for $x \neq x_0$. By definition of the product topology, this set is open. Moreover, since $x_0 \notin \{x \in [0,1] \mid U_{i,x} \neq [0,1]\}$, $U_{i,x_0} = [0,1]$ for all $i \in I$. So $\nexists\, U_i \subseteq H$. So $\{U_i\}_{i\in I}$ is not a countable neighborhood basis of 1, what is a contradiction. □

In addition to the "local complexity" property of first-countability, the "global complexity" property of second-countability helps us rule out metrizability in a more subtle way. Second-countability is stronger than first-countability and tells us that the topology of the space can be described with countably many open sets.

Definition 5 (Second-countable topological space)
*A topological space $(X, \mathcal{T})$ is **second-countable** if it has a countable basis.*

Lemma 4 (Second-countable ⇒ First-countable)
Any second-countable topological space is first-countable.

Proof.

Let $(X, \mathcal{T})$ be a second-countable topological space. Then it has a countable basis $\mathcal{B}$. For each $x \in X$, define $\mathcal{B}_x := \{B \in \mathcal{B} \mid x \in B\}$. First note, that for any x $\mathcal{B}_x$ is a subset of $\mathcal{B}$ and thus also countable. Now, let's show that for any $x \in X$, $\mathcal{B}_x$ is a neighborhood basis. Let $x \in X$. Take any U open, $U \in \mathcal{T}$ such that $x \in U$. Since $\mathcal{B}$ is a countable basis for $(X, \mathcal{T})$, U is the union of sets in $\mathcal{B}$. Since $x \in U$, one of the sets in this union contains x. Call this set B_x. $B_x \subseteq U$ and by definition of $\mathcal{B}_x$, $B_x \in \mathcal{B}_x$.

□

As mentioned above, second-countability helps us rule out metrizability for spaces in a more subtle way than before. For metrizable spaces, separability and second-countability must agree. More informally, for metrizable spaces, being "spread out" is the same as being "globally complex". So, whenever a space is separable, but not second-countable (or vice-versa), we can rule out metrizability.

Definition 6 (Separable topological space)
*Let $(X, \mathcal{T})$ be a topological space. $S \subseteq X$ is **dense** if $\overline{S} = X$. $(X, \mathcal{T})$ is **separable** if it has a countable dense subset.*

Lemma 5 (For metrizable spaces, separable ⇔ second-countable)
If $(X, \mathcal{T})$ is a metrizable topological space, then it is separable if and only if it is second-countable.

Proof.

First, let's show the right to left direction. Suppose that $(X, \mathcal{T})$ is a metrizable second-countable space. Then, it has a countable basis $\mathcal{B} = \{B_1, B_2, \dots\}$. We can assume without loss of generality that all sets in $\mathcal{B}$ are not empty. Take x_i from each B_i. I claim that $\{x_1, x_2, \dots\}$ is dense. Since it is countable, this implies that $(X, \mathcal{T})$ is separable. Let's show that it is indeed dense. By the limit-point characterization of closure, it suffices to show that for any open set $U \in \mathcal{T}$, $U \cap \{x_1, x_2, \dots\} \neq \varnothing$. Take any open set $U \in \mathcal{T}$. Since $\mathcal{B}$ is a basis, U is the union of some sets in $\mathcal{B}$. So there is some $B_i \in \mathcal{B}$ such that $B_i \subseteq U$. So $x_i \in U$. So $\{x_1, x_2, \dots\} \cap U \neq \varnothing$.

Now, let's show the left to right direction. Suppose that $(X, \mathcal{T})$ is a metrizable separable space. Then it has a dense subset $\{x_1, x_2, \dots\}$. And there is a metric d that induces $\mathcal{T}$. Let

$$\mathcal{U} = \left\{ B^d_{\frac{1}{n}}(x_m) \mid m, n \in \mathbb{N} \right\}$$

Note that $\mathcal{U}$ is countable. So it suffices to show that any open set $U \in \mathcal{T}$ is the union of sets in $\mathcal{U}$. Let $U \in \mathcal{T}$ be an open set. Since d induces $\mathcal{T}$, open d balls form a basis for $\mathcal{T}$. So $x \in B^d_r(p) \subseteq U$ for some real number $r > 0$ and some point p. Using r, the distance between k and x and the triangle inequality property of metrics, we can get $B^d_{r'}(x) \subseteq U$ for some real $r' > 0$. Pick n such that $2 < nr'$. Then $B^d_{\frac{2}{n}}(x) \subseteq U$. Since $\{x_1, x_2, \dots\}$ is dense, $B^d_{\frac{1}{n}}(x) \cap \{x_1, x_2, \dots\} \neq \varnothing$. In other words, $d(x_i, x) < \frac{1}{n}$ for some x_i. So $x \in B^d_{\frac{1}{n}}(x_i)$ and by the triangle inequality property of metrics, $B^d_{\frac{1}{n}}(x_i) \subseteq B^d_{\frac{2}{n}}(x)$. Thus $x \in B^d_{\frac{1}{n}}(x_i) \subseteq B^d_{\frac{2}{n}}(x) \subseteq U$. For any $x \in U$, we can construct such $B^d_{\frac{1}{n}}(x_i) \in \mathcal{U}$. Taking the union of all these sets, we get back U. $\square$

To exemplify how we can use second-countability to rule out metrizability, we can consider the Sorgenfrey line. This topological space is not amenable to any of the strategies to rule out metrizability previously employed: it is Hausdorff, regular and first-countable. It is, however, separable and not-second countable, so it must be not metrizable.

Example 5 (Sorgenfrey line is not metrizable)
The Sorgenfrey line is Hausdorff, regular and first-countable, but separable and not second-countable and thus not metrizable.

Proof.

The Sorgenfrey line is $\mathbb{R}$ equipped with the topology generated by $\{[a, b) \mid a, b \in \mathbb{R}\}$. Note first that it is Hausdorrf. Let x, y be distinct real numbers and assume without loss of generality that $x < y$. Then $[x, \frac{x+y}{2})$ and $[\frac{x+y}{2}, y+1)$ are disjoint basic open sets separating x and y.

In addition, the Sorgenfrey line is regular. Take any closed set C and any point $x \notin C$. Since $\{[a, b) \mid a, b \in \mathbb{R}\}$ generates the topology, $X \setminus C$ is the union of sets in this basis, one of which contains x, $[p, q)$. Note that $[p, q)$ is the complement of the union of all intervals $[a, b)$ with $b < p$ or $a > q$. So $[p, q)$ is closed and $X \setminus [p, q)$ is open. Since $[p, q) \subseteq X \setminus C$, $C \subseteq X \setminus [p, q)$. And $[p, q)$ and $X \setminus [p, q)$ are clearly disjoint. So $[p, q)$ and $X \setminus [p, q)$ are open sets separating x and C.

The Sorgenfrey line is also first-countable. Let $x \in \mathbb{R}$. I claim that

$$\mathcal{B}_x = \left\{ \left[x, x + \frac{1}{n}\right) \mid n \in \mathbb{N} \right\}$$

is a countable neighborhood basis of x. First, note that it is countable because $\mathbb{N}$ is countable. Now, let's show that for any open set U, there is some $B \in \mathcal{B}_x$ such that $B \subseteq U$. Let U be an open set. Then it is the union of basic open sets $[a, b)$, one of which contains x, $[w, y)$. So $[x, y) \subseteq [w, y) \subseteq U$. Pick a natural number n such that $1 < n(y - x)$. Then $\frac{1}{n} < q - y$ and $[x, x + \frac{1}{n}) \subseteq [x, y) \subseteq U$.

Next, we can show that the Sorgenfrey line is separable. We will show that $\mathbb{Q}$ is dense in the Sorgenfrey line. Since $\mathbb{Q}$ is countable, this suffices. Let's show that any nonempty open set intersects $\mathbb{Q}$. Take any nonempty open set U. Then it is the union of basic open sets $[a, b)$. Call one of these sets $[w, y)$. Since the rationals are dense in $\mathbb{R}$, there is a rational q between w and y. So $U \cap \mathbb{Q} \neq \varnothing$.

Finally, let's show that the Sorgenfrey line is not second-countable. We will show that any basis for the Sorgenfrey line must be uncountable. Let $\mathcal{B}$ be a basis the Sorgenfrey line. for For any $x \in \mathbb{R}$, $[x, x + 1)$ is an open set, so it is a union of sets in $\mathcal{B}$. One of these sets in $\mathcal{B}$ contains x. Call it B_x. So, we have $x \in B_x \subseteq [x, x + 1)$. Let y, z be distinct real numbers. Without loss of generality, suppose $y < z$. We have $y \in B_y$ and $B_z \subseteq [z, z + 1)$, so $y \in B_y$, but $y \notin B_z$. So for any two distinct real numbers y and z, B_y and B_z are distinct. Thus, since the real numbers are uncountable, $\mathcal{B}$ is also uncountable. □

The reader may have guessed by now that the reason we use second-countability to rule out metrizability more subtly is that the previous more straightforward method would not work. That is a good guess. There are metrizable spaces that are not second-countable - and thus not separable. For example, any uncountable set X with the discrete topology is metrizable, but not second-countable. So simply noting that a space is not second-countable does not suffice to rule out metrizability.

Example 6 (Any uncountable set X with the discrete topology is metrizable, but not second-countable) *Any uncountable set X with the discrete topology is metrizable, but not second-countable.*

Proof.

The discrete metric on X is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

By definition, for any $x \in X$, $B_1^d(x) = \{x\}$. So all singletons in the topology induced by d are open sets. Since finite intersections of open sets are open and unions of open sets are open, this implies that all subsets of X are open. In other words, the discrete metric induces the discrete topology on X. Now, a basis for the discrete topology must have every singleton since every singleton should be the union of basic open sets. So, since X is uncountable, any such basis must be uncountable. So X is metrizable, but not second-countable.

□

2 Sufficient conditions for metrizability

2.1 Urysohn's metrization theorem statement

Moving on from "intrinsic" topological properties that allow us to rule out metrizability, we have a natural new question. Is there a set of "intrinsic" topological properties that suffices for metrizability? Taking a first stab at this question, the reader may guess that some of the "intrinsic" topological properties we previously discussed do the job. Perhaps surprisingly, Hausdorffness, regularity and second-countability indeed suffice for metrizability. Intuitively, if a space has separated points, separated points and closed sets and is not globally complex, then it is metrizable. This result is Urysohn's metrization theorem.

Theorem 6 (Urysohn's metrization theorem)
Any regular Hausdorff second-countable topological space is metrizable.

By our discussion in the previous section, the following closely related result holds.

Proposition 7
$(X, \mathcal{T})$ is metrizable and separable if and only if $(X, \mathcal{T})$ is regular, Hausdorff and second-countable.

Proof.

By Urysohn's metrization theorem and lemma 5, the right to left direction holds. By lemma 1, lemma 2 and lemma 5, the left to right direction holds. □

We can use Urysohn's metrization theorem to show, for example, that topological manifolds, the basic objects of study in geometric topology, are metrizable. Let's work our way through some propositions using the property of compactness to get this result. Since topological manifolds are Hausdorff and second-countable by definition, we just need to show they are regular.

2.2 Topological manifolds are metrizable

Definition 7 (Topological manifolds)
*A topological space $(X, \mathcal{T})$ is a **topological manifold** if*

1. *$(X, \mathcal{T})$ is Hausdorff.*

2. *$(X, \mathcal{T})$ is second-countable.*

3. *$(X, \mathcal{T})$ is locally Euclidean, i.e. for any $x \in X$, there is an open set $U_x \in \mathcal{T}$ such that $x \in U$ and U_x with the subspace topology is homeomorphic to $\mathbb{R}^n$ is equipped with the standard topology.*

Definition 8 (Compact topological space)
*A topological space $(X, \mathcal{T})$ is **compact** if any of its open covers has a finite subcover.*

Definition 9 (Compact neighborhoods)
*A topological space $(X, \mathcal{T})$ has **compact neighborhoods** if for any point $x \in X$, there is a compact set $K \subseteq X$ such that $x \in K^\circ \subseteq K$.*

Proposition 8 (Locally Euclidean topological spaces have compact neighborhoods)
Any locally Euclidean topological space has compact neighborhoods.

Proof.

Let $(X, \mathcal{T})$ be a locally Euclidean topological space. Take any $x \in X$. Then there is an open set $U_x \in \mathcal{T}$ with $x \in U_x$ and a homeomorphism $\phi : U_x \to \mathbb{R}^n_{std}$. Let B be a closed ball of finite radius centered at $\phi(x)$. Note that $\phi(x) \in B^\circ$. So, $x \in \phi^{-1}(B^\circ) \subseteq \phi^{-1}(B)$. And since ϕ is a homeomorphism, $\phi^{-1}(B^\circ) = (\phi^{-1}(B))^\circ$. And since $\phi^{-1}(B)$ is the continuous image of a compact set, it is compact. So $x \in (\phi^{-1}(B))^\circ \subseteq \phi^{-1}(B)$, where $\phi^{-1}(B)$ is compact. □

Proposition 9 (Hausdorff topological spaces that have compact neighborhoods are regular)
Any Hausdorff topological space that has compact neighborhoods is regular.

Proof. Let $(X, \mathcal{T})$ be a Hausdorff topological space that has compact neighborhoods. Take any $x \in X$ and any closed set C such that $x \notin C$. Since $(X, \mathcal{T})$ has compact neighborhoods, there is a compact set $K \subseteq X$ such that $x \in K^\circ \subseteq K$. Since $(X, \mathcal{T})$ is Hausdorff, for every $y \in C \cap K$, we can get disjoint open sets U_y, V_y such that $x \in U_y$ and $y \in V_y$. Note that $C \cap K$ is compact because it is a closed set of the compact space K. So, since the V_y sets intersected with $C \cap K$ form an open cover of $C \cap K$, we have a finite subset of the V_y sets whose union contains $C \cap K$. Let this finite subset be $\{V_{y_1}, \ldots, V_{y_n}\}$.

Let

$$V := \bigcup_{i=1}^{n} V_{y_i} \text{ and } U := \bigcap_{i=1}^{n} U_{y_i}$$

As mentioned above, $C \cap K \subseteq V$. And since $x \in U_y$ for any $y \in C \cap K$, $x \in U$. Furthermore, since U_y and V_y are disjoint for any $y \in C \cap K$, V and U are disjoint. Since $(X, \mathcal{T})$ is Hausdorff and K is compact, it is also closed. Now, consider the sets $U \cap K^\circ$ and $V \cup (X \cap K)$. They are both unions of open sets, and thus open. Moreover, since $x \in U$ and $x \in K^\circ \subseteq K$, $x \in U \cap K^\circ$. And note that $C \setminus K \subseteq X \setminus K$ and from before $C \cap K \subseteq V$, so $C \subseteq V \cup (X \setminus K)$. Finally, note that $K^\circ \subseteq K$, $X \setminus K \subseteq X \setminus (K^\circ)$ and thus $K^\circ \cap (X \setminus K) = \varnothing$. And from before, U and V are disjoint. So the sets $U \cap K^\circ$ and $V \cup (X \cap K)$ are disjoint and separate x and C, as wanted.

□

Theorem 10 (Topological manifolds are metrizable)
Any topological manifold is metrizable.

Proof.

Since by definition topological manifolds are Hausdorff and second-countable, by Urysohn's metrization theorem it suffices to show that any topological manifold is regular. By propositions 8 and 9, topological manifolds are regular. □

2.3 Proving Urysohn's metrization theorem

Now, let's start to work towards a proof of Urysohn's metrization theorem. Below is the first important idea for our proof: metrizability is a topological property. It gives us some bearings as to what to do in the proof. A topological space homeomorphic to a metrizable space is metrizable. So we can construct a homeomorphism from an arbitrary Hausdorff, regular and second-countable space to some metrizable space to prove the theorem.

Lemma 11 (Metrizability is a topological property)
If a topological space is homeomorphic to a metrizable topological space, it is also metrizable.

Proof.

Let $(X, \mathcal{T}_X)$ be a topological space. Let $(Y, \mathcal{T}_Y)$ be a metrizable topological space. And let f be a homeomorphism from X to Y. Since $(Y, \mathcal{T}_Y)$ is a metrizable, some metric d_Y induces T_Y. Let's show that $d_X(a, b) = d_Y(f(a), f(b))$ is a metric that induces $\mathcal{T}_X$.

We get the first property of metrics by the injectivity of f:

$$d_X(a,b) = 0 \iff d_Y(f(a), f(b)) = 0 \iff f(a) = f(b) \iff a = b$$

We get the second and third property of metrics just by definition:

$$d_Y(f(a), f(b)) = d_Y(f(b), f(a)) \Rightarrow d_X(a,b) = d_X(b,a)$$

$$d_Y(f(a), f(b)) + d_Y(f(b), f(c)) \leq d_Y(f(a), f(c)) \Rightarrow d_X(a,b) + d_X(b,c) \leq d_X(a,c)$$

Now, let's show that any open set in $\mathcal{T}_X$ is a union of open d_X balls. Let U be an open set in $\mathcal{T}_X$. $f(U)$ is open, so it is a union of open d_Y balls $B^{d_Y}_{r_i}(y_i)$. And so $U = f^{-1}(f(U))$ is a union of sets

$$\{a \in X \mid d_Y(f(a), f(x_i)) < r_i\} = \{a \in X \mid d_X(a, x_i) < r_i\} = B^{d_X}_{r_i}(x_i)$$

$\square$

From our previous result, we saw that to prove Urysohn's metrization theorem we can construct a homeomorphism to a metrizable space. The next proposition is another ingredient in this direction for our proof. Countable products of metric spaces are metrizable and so our homeomorphism can be to a countable product of metric spaces. This is at least apparently promising because it suggests a more clear path in our proof.

Due to the universal property of product spaces, if we have a bunch of continuous maps to metric spaces, we can piece these maps together and get a continuous map to a countable product of metric spaces, which we now know is metrizable. Our hope is that maybe we can find continuous maps that "fit together nicely" and piece them together to get not only a continuous map to a metrizable space, but rather a homeomorphism to a metrizable space, which is all we need.

Lemma 12 (Equivalent bounded metric)
For any metrizable topological space $(X, \mathcal{T})$ there is a metric $d \leq 1$ that induces $\mathcal{T}$.

Proof.

Let $(X, \mathcal{T})$ be a metrizable topological space and let $f : X \times X \to X$ be a metric that induces $\mathcal{T}$. Let's show that $d : X \times X \to X$ given by $d(x,y) = \min(f(x,y), 1)$ is a metric that induces $\mathcal{T}$. We start by showing that d is a metric.
First,

$$d(x,y) = 0 \iff \min(f(x,y), 1) \iff f(x,y) = 0 \iff x = y$$

For symmetry,

$$f(x,y) = f(y,x) \Rightarrow \min(f(x,y), 1) = \min(f(y,x), 1) \iff d(x,y) = d(y,x)$$

And as for the triangle inequality,

$$\min(f(x,y), 1) + \min(f(y,z), 1) = \min(f(x,y) + f(y,z), f(x,y) + 1, f(y,z) + 1, 2)$$

By the triangle inequality for f,

$$f(x,y) + f(y,z) \geq f(x,z) \geq \min(f(x,z), 1)$$

And

$$f(x,y)+1, f(y,z)+1, 2 \geq 1 \geq \min(f(x,z),1)$$

So

$$d(x,y)+d(x,z)=\min(f(x,y),1)+\min(f(y,z),1) \geq \min(f(x,z),1)=d(x,z)$$

Now, let's show that d induces $\mathcal{T}$.
Note that for $r \leq 1$,

$$d(x,y)=\min(f(x,y),1) \leq r \iff f(x,y) \leq r$$

In other words, $B_r^d(x)=B_r^f(x)$ for $r \leq 1$.
Take any open ball $B_r^f(x)$ with arbitrary r. Take any y in this open ball.

$$B^f_{\min(1,r-f(x,y))}(y) \subseteq B^f_r(x) \text{ and } B^f_{\min(1,r-f(x,y))}(y) = B^d_{\min(1,r-f(x,y))}(y)$$

So, open f-balls are locally d-open and thus d-open.
Now, take any open ball $B_r^d(x)$ with arbitrary r. Take any y in this open ball. Note that for any $r \geq 0$,

$$d(x,y)=\min(f(x,y),1) \leq r \Rightarrow f(x,y) \leq r$$

So, $B^f_{r-d(x,y)}(y) \subseteq B^d_{r-d(x,y)}(y) \subseteq B^d_r(x)$. Thus, open d-balls are locally f-open and thus f-open. So we can conclude that f and d induce the same topology, $\mathcal{T}$.

$\square$

Proposition 13 (Countable products of metric spaces are metrizable)
Any countable product of metric spaces is metrizable.

Proof.

Let (X_i, d_i) for $i \in \mathbb{N}$ be metric spaces. By the previous lemma, we can assume that $d_i \leq 1$ for $i \in \mathbb{N}$.
Let's show that

$$d(x,y)=\sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i,y_i)$$

is a metric that induces the product topology on $\prod\limits_{i=1}^{\infty} X_i$.
First, let's show that d is a metric.

$$d(x,y)=0 \iff \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i,y_i)=0 \iff d_i(x_i,y_i)=0 \text{ for all } i \iff x_i=y_i \text{ for all } i \iff x=y$$

$$d(x,y)=\sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i,y_i)=\sum_{i=1}^{\infty} \frac{1}{2^i} d_i(y_i,x_i)=d(y,x)$$

$$d(x,y)+d(y,z)=\sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i,y_i)+\sum_{i=1}^{\infty} \frac{1}{2^i} d_i(y_i,z_i)=\sum_{i=1}^{\infty} \frac{1}{2^i}[d_i(x_i,y_i)+d_i(y_i,z_i)] \leq \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i,z_i)=d(x,z)$$

Now, let's show that d induces the product topology on $\prod X_i$. First, we will show that open sets in the product topology are open in the $d-$topology. Let $O := \prod O_i$ be an open set in the product topology on $\prod X_i$.

By the definition of product topology, there is a finite set $D \subseteq \mathbb{N}$ such that for any $k \in D$, $O_k \neq X_k$. Pick $x \in O$. For each $k \in D$, since O_k is open in X_k, there is an open ball $B^{d_k}_{r_k}(x_k) \subseteq O_k$. Since D is finite, we can get $0 < r < 1$ such that $r \leq \frac{r_k}{2^k}$ for all $k \in D$. We will see that $B^d_r(x) \subseteq O$.
Suppose $y \in B^d_r(x), d(x,y) < r$. Then for any $k \in D$,

$$\frac{d_k(x_k, y_k)}{2^k} \leq d(x,y) \text{ by definition of } d$$

and

$$d(x,y) < r \leq \frac{r_k}{2^k} \text{ by assumption and the definition of } r$$

So, for any $k \in D$, $d_k(x_k, y_k) < r_k$. Since from above $B^{d_k}_{r_k}(x_k) \subseteq O_K$, $y_k \in O_k$. And for O_i with indices outside D, since O_i is equal to X_i, $y_i \in O_i$. So $y \in O$. And since y was arbitrary, $B^d_r(x) \subseteq O$. So, open sets in the product topology are locally $d-$open and thus $d-$open.

Now, let's show that $d-$open sets are open in the product topology. Let U be a $d-$open set. Pick $x \in U$. Since U is $d-$open, $B^d_r(x) \subseteq U$. The goal here is to construct an open set O in the product topology with $x \in O \subseteq B^d_r(x)$ to show that U is locally open with respect to the product topology. The main idea to build O is that the tail in the sum defining d does not matter. We start by picking the point at which it this tail starts.

Let $t \in \mathbb{N}$ be a natural number such that $\frac{1}{2^t} < \frac{r}{2}$. Construct $O := \prod O_i$ by letting $O_k = B^{d_k}_{\frac{r}{2t}}(x)$ for $1 \leq k \leq t$ and letting $O_k = X_k$ for $k > t$. Clearly, $x \in O$. Now, let's show that $O \subseteq B^d_r(x)$. Suppose $y \in O$. Then, by definition of O for $1 \leq k \leq t$, $d_k(x_k, y_k) < \frac{r}{2t}$. So,

$$\sum_{i=1}^{t} \frac{1}{2^i} d_i(x_i, y_i) \leq \sum_{i=1}^{t} d_i(x_i, y_i) < t \times \frac{r}{2t} = \frac{r}{2}$$

And since $d_i \leq 1$ and by our choice of t,

$$\sum_{i=t+1}^{\infty} \frac{1}{2^i} d_i(x_i, y_i) \leq \sum_{i=t+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^t} < \frac{r}{2}$$

So, $d(x,y) < \frac{r}{2} + \frac{r}{2} = r$, i.e. $y \in B^d_r(x)$. So, $x \in O \subseteq B^d_r(x) \subseteq U$. So, U is locally open in the product topology and thus open in the product topology. Since U was arbitrary, $d-$open sets are open in the product topology. □

Last time we left our reflection on how to prove Urysohn's metrization theorem with the hope that we can find continuous maps to metric spaces that "fit together nicely" giving us not only a continuous map, but rather a homeomorphism to a metrizable space. Such maps really do exist. Urysohn's lemma gives us maps that will "fit together nicely" in our proof. But first, since Urysohn's lemma applies to normal spaces, we need to show that any Hausdorff, regular, second-countable space is normal.

Definition 10 (Lindelöf topological space)
*A topological space $(X, \mathcal{T})$ is **Lindelöf** if every open cover of $(X, \mathcal{T})$ has a countable subcover.*

Lemma 14 (Second-countable $\Rightarrow$ Lindelöf)
Any second-countable topological space is Lindelöf.

Proof.

Let X be a second-countable space. Let $\mathcal{U}$ be an open cover of X. Since X is second-countable, it has a countable basis $\{B_i\}_{i\in I}$. Since $\{B_i\}_{i\in I}$ is a basis of X, every $U \in \mathcal{U}$ is equal to $\bigcup_{j\in J_U} B_j$ for some $J_U \subseteq I$.

Let $\mathcal{K} := \{B_j \mid j \in J_U \text{ for some } U\}$. By definition of $\mathcal{K}$, for every $B \in K$, there is some $U_B \in \mathcal{U}$ such that $B \subseteq U_B$. For every $B \in \mathcal{K}$ assign one such U_B. Let $\mathcal{U}^* := \{U \in \mathcal{U} \mid U = U_B \text{ for some } B \in \mathcal{K}\}$. By definition of $\mathcal{U}^*$, for any $B \in \mathcal{K}$, $B \subseteq U$ for some U^* in $\mathcal{U}^*$. So,

$$\bigcup_{B\in\mathcal{K}} B \subseteq \bigcup_{U^*\in\mathcal{U}^*} U^*$$

Also, by definition of $\mathcal{U}^*$, $\mathcal{U}^* \subseteq \mathcal{U}$, so

$$\bigcup_{U^*\in\mathcal{U}^*} U^* \subseteq \bigcup_{U\in\mathcal{U}} U$$

And by the definition of $\mathcal{K}$,

$$\bigcup_{B\in\mathcal{K}} B = \bigcup_{U\in\mathcal{U}} U$$

So,

$$\bigcup_{U^*\in\mathcal{U}^*} U^* = \bigcup_{U\in\mathcal{U}} U = X$$

So, $\mathcal{U}^*$ is a subcover of $\mathcal{U}$. Moreover, since $\{B_i\}_{i\in I}$ is countable, so is $\mathcal{K}$ and in turn so is $\mathcal{U}^*$. So $\mathcal{U}^*$ is a countable subcover of $\mathcal{U}$. Since $\mathcal{U}$ was arbitrary, this shows that X is Lindelöf. $\square$

Definition 11 (Normal topological space)
A topological space $(X, \mathcal{T})$ is ***normal*** *if for any two disjoint closed sets A, B there exist open disjoint sets $U, V \in \mathcal{T}$ such that $A \subseteq U$ and $B \subseteq V$.*

Lemma 15 (Tychonoff's Lemma)
Any regular Lindelöf space is normal.

Proof.

Let $(X, \mathcal{T})$ be a regular Lindelöf topological space. Let A, B be two disjoint closed sets in $(X, \mathcal{T})$. Since $(X, \mathcal{T})$ is regular, for any $x \in A$, there exist open sets $U_x, Q_x \in \mathcal{T}$ such that $x \in U_x$ and $B \subseteq Q_x$. And since $U_x \cap Q_x = \varnothing$, U_x is a subset of the complement of Q_x, $U_x \subseteq Q_x^c$. And since $B \subseteq Q_x$, $Q_x^c \subseteq B^c$. So, $U_x \subseteq \overline{U_x} \subseteq \overline{Q_x^c} = Q_x^c \subseteq B^c$.

Note that $A \subseteq \bigcup_{x\in A} U_x$. So $X = A^c \cup \bigcup_{x\in A} U_x$ and $\{A^c\} \cup \{U_x\}_{x\in A}$ is an open cover of X. Since X is Lindelöf, $\{A^c\} \cup \{U_x\}_{x\in A}$ has a countable subcover. So, $A \subseteq \bigcup_{n\in\mathbb{N}} U_n$ where U_n are open and $\overline{U_n} \subseteq B^c$.

By the same argument exchanging the roles of A and B, $B \subseteq \bigcup_{n\in\mathbb{N}} V_i$ where V_n are open and $\overline{V_n} \subseteq A^c$. At this point, we have open covers of A and B, but we cannot guarantee that they are disjoint, as we would like. So we define

$$O_n := U_n \setminus \bigcup_{j=1}^{n} \overline{V_i} = U_n \cap \bigcap_{j=1}^{n} \overline{V_i}^c$$

$$W_n := V_n \setminus \bigcup_{i=1}^{n} \overline{U_i} = V_n \cap \bigcap_{i=1}^{n} \overline{U_i}^c$$

Note that O_n and W_n are open because they are finite intersections of open sets. We know that for each V_i, $\overline{V_i} \subseteq A^c$, so $A \subseteq \overline{V_i}^c$. So, $A \subseteq \bigcap\limits_{i=1}^{n} \overline{V_i}^c$. So, $A \subseteq U_i \cap \bigcap\limits_{i}^{n} \overline{V_i}^c = O_i$ for any i. So $A \subseteq \bigcup\limits_{n\in\mathbb{N}} O_n$.

By a similar argument, $B \subseteq \bigcup\limits_{n\in\mathbb{N}} W_n$.

Define $O := \bigcup\limits_{n\in\mathbb{N}} O_n$ and $W := \bigcup\limits_{n\in\mathbb{N}} W_n$.

Note that O and W are open because they are unions of open sets.

Now, let's show that O and W are disjoint.

$$O \cap W = \bigcup_{n\in\mathbb{N}} O_n \cap \bigcup_{n\in\mathbb{N}} W_n = \bigcup_{(m,n)\in\mathbb{N}^2} O_m \cap W_n$$

If $i \geq j$, $O_i \cap W_j = \left(U_i \cap \bigcap\limits_{k=1}^{i} \overline{V_k}^c\right) \cap \left(V_j \cap \bigcap\limits_{k=1}^{j} \overline{U_k}^c\right) \subseteq \overline{V_j}^c \cap V_j \subseteq V_j^c \cap V_j = \varnothing$.

If $j \geq i$, $O_i \cap W_j = \left(U_i \cap \bigcap\limits_{k=1}^{i} \overline{V_k}^c\right) \cap \left(V_j \cap \bigcap\limits_{k=1}^{j} \overline{U_k}^c\right) \subseteq U_i \cap \overline{U_i}^c \subseteq U_i \cap U_i^c = \varnothing$.

So $O \cap W = \varnothing$. □

Proposition 16 (Urysohn's Lemma)
If A and B are disjoint closed subsets of a normal topological space $(X, \mathcal{T})$, then there exists a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Proof.

Let $(X, \mathcal{T})$ be a normal topological space. Let A, B be disjoint closed subsets of $(X, \mathcal{T})$. Let D be the set of dyadic rationals. We will build a collection of open sets $\{U(d) \mid d \in D\}$ inductively such that:

1. for any $d \in D$, $U(d)$ is open (except maybe for $U(0)$) and $A \subseteq U(d) \subseteq X \setminus B$.
2. if $d_1 < d_2$, $\overline{U(d_1)} \subseteq U(d_2)$.

Define $D_n := \left\{\frac{k}{2^n} \mid k \in \{0, \dots, 2^n\}\right\}$.

Base case: construct $U(d)$ for $d \in D_0$

$U(0) := A$, $U(1) := X \setminus B$. Note that $U(1)$ is open because B is closed. And $\overline{A} = A$ since A is closed, so property 2 is satisfied.

Inductive step: suppose we have $U(d)$ for $d \in D_n$, construct for $d \in D_{n+1}$.

For each $k \in \{0, \dots, 2^n\}$, proceed as follows to get $U(\frac{2k+1}{2^{n+1}})$. By property 2, we have

$$\overline{U(k/2^n)} = \overline{U(2k/2^{n+1})} \subseteq U(2k+2/2^{n+1}) = U(k+1/2^n)$$

Since $\overline{U(\frac{k}{2^n})}$ is closed and $X \setminus U(\frac{k+1}{2^n})$ is closed, by normality there exist disjoint open sets U, V such that $\overline{U(\frac{k}{2^n})} \subseteq U$ and $X \setminus U(\frac{k+1}{2^n}) \subseteq V$. Since $X \setminus U(\frac{k+1}{2^n}) \subseteq V$, $X \setminus V \subseteq U(\frac{k+1}{2^n})$. And since $U \cap V = \varnothing$, $U \subseteq X \setminus V$. So $\overline{U(\frac{k}{2^n})} = \overline{U(\frac{2k}{2^{n+1}})} \subseteq U \subseteq X \setminus V \subseteq U(\frac{2k+2}{2^{n+1}}) = U(\frac{k+1}{2^n})$. Moreover, since $U \subseteq X \setminus V$ and $X \setminus V$ is closed,

$U \subseteq \overline{U} \subseteq X \setminus V$. So $\overline{U(\frac{k}{2^n})} = \overline{U(\frac{2k}{2^{n+1}})} \subseteq U \subseteq \overline{U} \subseteq U(\frac{2k+2}{2^{n+1}}) = U(\frac{k+1}{2^n})$. And we can define $U(\frac{2k+1}{2^{n+1}}) := U$ since U satisfies properties 1 and 2.

Now we are ready to construct the continuous function f. Define $f : X \to [0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \notin U_1 \\ \inf\{d \in D \mid x \in U(d)\} & \text{otherwise} \end{cases}$$

Note that $f(A) = \{0\}$ because $U_0 = A$ and $f(B) = \{1\}$ because $U_1 = X \setminus B$. Now, we need to show that f is continuous. First, note that

1. If $r \in D$ and $x \in U(r)$, then $f(x) \leq r$.

   That's because $r \in D$ and $x \in U(r)$ means that r is a member of $\{d \in D \mid x \in U(d)\}$ and so the infimum of this set, $f(x)$, is below r.

2. If $r \in D$ and $x \notin \overline{U(r)}$, then $f(x) \geq r$.

   That's because $U(d_1) \subseteq \overline{U(d_1)} \subseteq U(d_2) \subseteq \overline{U(d_2)}$ whenever $d_1 < d_2$. So $x \notin U(r) \Rightarrow x \notin U(d)$ for $d < r$. So if $r \in D$ and $x \notin \overline{U(r)}$, we know that there are no members smaller than r in $\{d \in D \mid x \in U(d)\}$, so r is a lower bound of the set. So $\inf\{d \in D \mid x \in U(d)\} \geq r$.

Let $x \in X$ and $\varepsilon > 0$.

- If $f(x) = 0$, choose $r \in D$ such that $0 < r < \varepsilon$. Since $f(x) = 0$, $x \in U(r)$. And by property 1 above, $f(U(r)) \subseteq [0, \varepsilon)$.
- If $f(x) = 1$, choose $r \in D$ such that $1 - \varepsilon < r < 1$. Since $f(x) = 1, x \notin U(1)$. So, by property 2 of our $U(d)$ open sets, $x \notin \overline{U(r)}$. And by property 2 just above, $f(X \setminus \overline{U(r)}) \subseteq (1 - \varepsilon, 1]$.
- If $0 < f(x) < 1$, choose $r, r', s \in D$ such that

  $$f(x) - \varepsilon < r' < r < f(x) < s < f(x) + \varepsilon$$

  Since $f(x) > r$, $x \notin U_r$, so by property 2 of $U(d)$, $x \notin \overline{U(r')}$. And since $f(x) < s$, $\exists k \in D, k < s$ such that $x \in U(k)$. So, by property 2 of our $U(d)$ open sets, $x \in U(k) \subseteq \overline{U(k)} \subseteq U(s)$. So $x \in U(s)$. So $x \in U(s) \setminus \overline{U(r')}$. And by properties 1 and 2 just above, $f(x) - \varepsilon < r' \leq f(U(s) \setminus \overline{U(r')}) \leq s < f(x) + \varepsilon$. So, $f(U(s) \setminus \overline{U(r')}) \subseteq (f(x) - \varepsilon, f(x) + \varepsilon)$.

□

Now, we have almost all the tools necessary to prove Urysohn's metrization theorem. The only thing left is to clarify why the maps we get by using Urysohn's lemma "fit together nicely." Two factors work together to make this the case. The most important of these factors is that the maps we get by using Urysohn's lemma "distinguish between points and closed sets." And the secondary factor contributing to the maps "fitting together nicely" is that the domain of these maps is Hausdorff. Let's prove the theorem and see this explictly!

Proof of Urysohn's metrization theorem.

Let $(X, \mathcal{T})$ be a regular Hausdorff second-countable topological space. Let $\mathcal{B}$ be a countable basis of $\mathcal{T}$. Define $\mathcal{A}$, the set of pairs (U, V) such that $U, V \in \mathcal{B}$ and $\overline{U} \subseteq V$. Since $\mathcal{B}$ is countable, so is $\mathcal{A}$.

By lemma 8, 9 and Urysohn's lemma, for each pair $(U, V) \in \mathcal{A}$ there exists a continuous map $f_{(U,V)} : X \to [0,1]$ such that $f(\overline{U}) = \{0\}$ and $f(X \setminus V) = \{1\}$. Let $\mathcal{F}$ be the family of such functions f. Define $E : X \to [0,1]^{\mathbb{N}}$ to be the function whose components are the functions in $\mathcal{F}$. By the universal property of the product topology, E is continuous.

We will use the following claim to show that E is an embedding.

Claim: F distinguishes points and closed sets, i.e. for any closed set $B \in (X, \mathcal{T})$ and any $x \in X \setminus B$, there is some f in the family $\mathcal{F}$ such that $f(x) \notin \overline{f(B)}$.

Proof of claim:
Let B be a closed set in $(X, \mathcal{T})$. Choose $V \in \mathcal{B}$ such that $x \in V \subseteq X \setminus B$. Since $(X, \mathcal{T})$ is regular and $X \setminus V$ is closed and $x \in V$, we get O, W open disjoint such that $X \setminus V \subseteq O$ and $x \in W$. Since $O \cap W = \varnothing$, $W \subseteq X \setminus O \subseteq V$ and $X \setminus O$ is closed, $W \subseteq \overline{W} \subseteq X \setminus O \subseteq V$.

Pick $U \in \mathcal{B}$ such that $x \in U \subseteq W$. Since $U \subseteq W$, $\overline{U} \subseteq \overline{W} \subseteq V$. Consider the function f in $\mathcal{F}$ chosen for the pair (U, V). $f(x) = 0$, since $x \in U \subseteq \overline{U}$ and $f(\overline{U}) = \{0\}$. And since $f(X \setminus V) = \{1\}$ and $B \subseteq X \setminus V$, $f(B) = \{1\}$. So, since points are closed in any Hausdorff space and $[0,1]$ is Hausdorff, $f(B) = \overline{f(B)} = \{1\}$. So $f(x) \notin \overline{f(B)}$.

This finishes the proof of the claim.

Back to the main line of argument in the proof, let's show that E is injective. Since $(X, \mathcal{T})$ is Hausdorff, points are closed in $(X, \mathcal{T})$. So, by our claim above, for any distinct $x, y \in X$, there is some f in the family $\mathcal{F}$ such that $f(x) \notin \overline{f(\{y\})}$. So $f(x) \neq f(y)$.

Since E is continuous and injective, restricting E to its image, we get a continuous, injective and surjective map H between X and a subspace of $[0,1]^{\mathbb{N}}$. Now, let's show that H is open. Take any open set U in $(X, \mathcal{T})$. We will show that $H(U)$ is locally open and thus open. Take any point $y \in H(U)$. There is $x \in U$ such that $H(x) = y$. Since $\mathcal{F}$ distinguishes points and closed sets, there is some f such that $f(x) \notin \overline{f(X \setminus U)}$. By definition of the product topology on $[0,1]^{\mathbb{N}}$, $Z := \{z \in [0,1]^{\mathbb{N}} \mid z_f \notin \overline{f(X \setminus U)}\}$ is open. So $H(X) \cap Z$ is open in the subspace topology on $H(X)$. And $z_f \notin \overline{f(X \setminus U)} \Rightarrow z_f \notin f(X \setminus U) \Rightarrow z \notin H(X \setminus U) \Rightarrow z \in H(U)$. So $H(X) \cap Z \subseteq H(U)$.

Since subspaces of metrizable spaces are metrizable, by lemma 5 X is metrizable.

□

3 Equivalent characterization of metrizability

3.1 Urysohn metrization theorem fails

Urysohn's metrization theorem is an astouding result. But one may still ask if it leaves any room for improvement. Urysohn's metrization theorem gives us a set of "intrinsic" topological properties that suffice for metrizability. Could we get a set of "intrinsic" topological properties that is equivalent to metrizability?

Looking back at the last example in section 1, we know that not all metrizable spaces are second-countable. So the "intrinsic" topological properties in Urysohn's metrization theorem, which include second-countability, are clearly not equivalent to metrizability. There are metrizable spaces that Urysohn's metrization theorem fails to notice such as any uncountable discrete space. However, we can hope to change second-countability to another "complexity" property and get a set of "intrinsic" topological properites equivalent to metrizability. Happily, this works. We can substitute the property of having a $\sigma-$locally finite basis for the property of second-countability and get a characterization of metrizability in terms of "intrinsic" topological properties. This is the Nagata-Smirnov metrization theorem.

3.2 Nagata-Smirnov metrization theorem

Definition 12 (Locally finite collection, $\sigma-$locally finite collection)
*Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{S}$ of subsets $S \subseteq X$ is **locally finite** if for any $x \in X$, there is an open set $U \in \mathcal{T}$ such that $U \cap S \neq \varnothing$ only for finitely $S \in \mathcal{S}$. A collection $\mathcal{S}$ of subsets $S \subseteq X$ is $\sigma-$**locally finite** if it is the countable union of locally finite collections.*

Theorem 17 (Nagata-Smirnov metrization theorem)
A topological space X is metrizable if and only if it is Hausdorff and regular and has a σ-locally finite basis.

Our proof for the right-to-left direction of this theorem is somewhat similar to our proof for Urysohn's metrization theorem. We start by showing that any Hausdorff, regular space that has a σ-locally finite basis is normal. This is a variant of Tychonoff's lemma. This variant of Tychonoff's lemma will allow us to use continuous maps given by Urysohn's lemma to create other continuous maps. Finally, we will use these new continuous maps to build an embedding to a space whose underlying set is a product of $[0, 1]$ and whose topology is a special metric topology. Restricting the codomain of this embedding, we get a homeomorphism to a metrizable space, which is again sufficient for our purposes.

Lemma 18 (Closure of union of sets in locally finite collection)
If $\mathcal{S}$ is a locally finite collection of a topological space $(X, \mathcal{T})$, then $\overline{\bigcup\limits_{S \in \mathcal{S}} S} = \bigcup\limits_{S \in \mathcal{S}} \overline{S}$.

Proof.

Let $Y := \bigcup\limits_{S \in \mathcal{S}} S$. First, note that $\bigcup\limits_{S \in \mathcal{S}} \overline{S} \subseteq \overline{Y}$. For each $S \in \mathcal{S}$, $S \subseteq \bigcup\limits_{S \in \mathcal{S}} S = Y$, so $\overline{S} \subseteq \overline{Y}$. Now, let's show that $\overline{Y} \subseteq \bigcup\limits_{S \in \mathcal{S}} \overline{S}$. Let $x \notin \bigcup\limits_{S \in \mathcal{S}} \overline{S}$. By the local finiteness of $\mathcal{S}$, there is an open set $U \in \mathcal{T}$ such that $x \in \mathcal{U}$ and there only finitely many sets $S_1, \ldots, S_k$ such that $S_i \cap U \neq \varnothing$.

Define

$$W := U \cap \left(X \setminus \bigcup_{i=1}^{k} \overline{S_i} \right)$$

W is open because it is the intersection of two open sets. Moreover, it is disjoint from the finitely many S_i since $S_i \subseteq \overline{S}_i$. And it is disjoint from the other sets $S \in \mathcal{S}$ because U is. And since $x \notin \bigcup\limits_{S \in \mathcal{S}} \overline{S}$ and $x \in U$, x belongs to W. So W is an open neighborhood of x that does not meet Y. Thus, by the limit-point characterization of closure, $x \notin \overline{Y}$. □

Proposition 19 (Tychonoff's Lemma Variant)
Any regular topological space that has a σ-locally finite basis is normal.

Proof.

Let $(X, \mathcal{T})$ be a regular topological space that has a $\sigma-$locally finite basis $\mathcal{B}$.
Claim: For any open set $W \in \mathcal{T}$, $W = \bigcup_{n\in\mathbb{N}} U_n = \bigcup_{n\in\mathbb{N}} \overline{U_n}$ with U_n open for all $n \in \mathbb{N}$.

Proof of claim:
Since $\mathcal{B}$ is $\sigma-$locally finite, $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$, where $\mathcal{B}_n$ is locally finite for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define

$$\mathcal{C}_n := \{B \in \mathcal{B}_n \mid \overline{B} \subseteq W\}$$

Since $\mathcal{C}_n \subseteq \mathcal{B}_n$, $\mathcal{C}_n$ is also locally finite for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define $U_n := \bigcup_{B\in\mathcal{C}_n} B$. Any member of $\mathcal{C}_n$ for some $n \in \mathbb{N}$ is a member of $\mathcal{B}$, and so a basic open set. So for each $n \in \mathbb{N}$, U_n is a union of basic open sets, and thus open. Since $\mathcal{C}_n$ is locally finite, by the lemma above, $\overline{U_n} = \bigcup_{B\in\mathcal{C}_n} \overline{B}$. Moreover, by the definition of $\mathcal{C}_n$, $\overline{B} \subseteq W$ for any $B \in \mathcal{C}_n$, so $\overline{U_n} = \bigcup_{B\in\mathcal{C}_n} \overline{B} \subseteq W$.

Now, it suffices to show that $W \subseteq \bigcup_{n\in\mathbb{N}} U_n$, which is what we will do. Let $x \in W$. Then, by the regularity of $(X, \mathcal{T})$ there are disjoint open sets U, V such that $x \in U$ and $X \setminus W \subseteq V$. So

$$x \in U \subseteq X \setminus V \subseteq W$$

Since $\mathcal{B}$ is a basis, U is the union of sets in $\mathcal{B}$. One of these sets contains x. So we have $x \in B \subseteq U \subseteq X \setminus V$ for some $B \in \mathcal{B}$. Since $B \subseteq X \setminus V$, $\overline{B} \subseteq \overline{X \setminus V}$. So, since $X \setminus V$ is closed, we have

$$\overline{B} \subseteq \overline{X \setminus V} = X \setminus V \subseteq W$$

Since $B \in \mathcal{B}$, $B \in \mathcal{B}_n$ for some $n \in \mathbb{N}$. So, since $\overline{B} \subseteq W$, by the definition of $\mathcal{C}_n$,

$$x \in B \subseteq \bigcup_{B\in\mathcal{C}_n} B = U_n \subseteq \bigcup_{n\in\mathbb{N}} U_n$$

This finishes the proof of the claim.

Back to the main line of argument in the proof, let's show that $(X, \mathcal{T})$ is normal. Let A, B be two closed sets in the topological space. Since $X \setminus B$ is open, by the claim above there exist open sets U_n such that

$$X \setminus B = \bigcup_{n\in\mathbb{N}} U_n = \bigcup_{n\in\mathbb{N}} \overline{U_n}$$

Trivially, for each $n \in \mathbb{N}$, $\overline{U_n} \subseteq X \setminus B$. Moreover, since $A \cap B = \varnothing$, $A \subseteq X \setminus B = \bigcup_{n\in\mathbb{N}} U_n$.
By the same reasoning, we get open sets V_n such that for each $n \in \mathbb{N}$, $\overline{V_n} \subseteq X \setminus A$ and $B \subseteq \bigcup_{n\in\mathbb{N}} U_n$. Now, we are exactly in the same position as in the middle of the proof of Tychonoff's lemma: we have open covers of A and B with the property that the closures of the sets in the open cover of one set do not meet the other set. Exactly the same procedure as in the proof of Tychonoff's lemma gives us disjoint open covers of A and B, as we want. $\square$

Now, let's prove the right-to-left direction of the Nagata-Smirnov theorem.

Proof of right-to-left direction of the Nagata-Smirnov theorem.

Let $(X, \mathcal{T})$ be a Hausdorff and regular topological space that has a $\sigma-$locally finite basis. Let's first talk about the metric topology that the codomain of our homeomorphism will have. Let $\mathcal{B}$ be a $\sigma-$locally finite basis of the topological space. The codomain of our homeomorphism is the set $[0,1]^{\mathcal{B}}$ with the metric d defined by

$$d(x,y) = \sup_{B\in\mathcal{B}}\{|x_B - y_B|\}$$

Let's show that this is a metric. First note that it is nonnegative because for any real number r, $|r| \geq 0$, so in particular $|x_B - y_B| \geq 0$ and and so $\sup_{B\in\mathcal{B}}\{|x_B - y_B|\} \geq 0$. Next, note that if $d(x,y) = 0$, then for every $B \in \mathcal{B}$, $0 \geq |x_B - y_B| \geq 0$, so $|x_B - y_B| = 0$ and since $x_B, y_B \geq 0$, $x_B = y_B$. So if $d(x,y) = 0$, $x = y$. Conversely, if $x = y$, for all $B \in \mathcal{B}$, $x_B = y_B$, so $|x_B - y_B| = 0$. So 0 is an upper bound of $\{|x_B - y_B|\}_{B\in\mathcal{B}}$. So $d(x,y) \leq 0$. And from above, $d(x,y) \geq 0$. So $d(x,y) = 0$. The map d is also symmetric because $|r| = |-r|$ for any real number r:

$$d(x,y) = \sup_{B\in\mathcal{B}}\{|x_B - y_B|\} = \sup_{B\in\mathcal{B}}\{|y_B - x_B|\} = d(y,x)$$

Finally, d satisfies the triangle inequality. For sets A, B of real numbers $\sup(A) + \sup(B) = \sup(A + B)$, where $A + B := \{a + b \mid a \in A, b \in B\}$. For our particular case, $\sup_{B\in\mathcal{B}}\{|x_B - y_B|\} + \sup_{B\in\mathcal{B}}(|y_B - z_B|) = \sup_{B\in\mathcal{B}}(|x_B - y_B| + |y_B - z_B|)$. And since the standard absolute value on $\mathbb{R}$ satisfies the triangle inequality, $|x_B - z_B| \leq |x_B - y_B| + |y_B - z_B|$ for all $B \in \mathcal{B}$. So upper bounds of $\{|x_B - y_B| + |y_B - z_B|\}_{B\in\mathcal{B}}$ are a proper subset of the upper bounds of $\{|x_B - z_B|\}_{B\in\mathcal{B}}$. So, $\sup_{B\in\mathcal{B}}(|x_B - z_B|) \leq \sup_{B\in\mathcal{B}}\{|x_B - y_B| + |y_B - z_B|\}$. Putting all of this together,

$$d(x,z) = \sup_{B\in\mathcal{B}}(|x_B - z_B|) \leq \sup_{B\in\mathcal{B}}\{|x_B - y_B| + |y_B - z_B|\} = \sup_{B\in\mathcal{B}}\{|x_B - y_B|\} + \sup_{B\in\mathcal{B}}(|y_B - z_B|) = d(x,y) + d(y,z)$$

Now, let's create new special continuous maps using Urysohn's lemma that we will later use to make the embedding we want.

Claim: For any open set $W \in \mathcal{T}$, there exists a continuous map $f : X \to [0,1]$ such that $0 \notin f(W)$ and $f(X \setminus W) = \{0\}$.

Proof of claim:
By the claim in proposition 19, $W = \bigcup_{n\in\mathbb{N}} U_n = \bigcup_{n\in\mathbb{N}} \overline{U_n}$, where each U_n is an open set. $W \cap X \setminus W = \varnothing$, so $\overline{U_n} \cap X \setminus W = \varnothing$ for each $n \in \mathbb{N}$. By the variant of Tychonoff's lemma above, $(X, \mathcal{T})$ is normal, so we can apply Urysohn's lemma to get for each U_n a map $f_n : X \to [0,1]$ satisfying $f_n(\overline{U_n}) = \{1\}$ and $f_n(X \setminus W) = \{0\}$. Define $f : X \to [0,1]$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n}$$

This function is well-defined because for any $x \in X$, $0 \leq f_n(x) \leq 1$, so $0 \leq \frac{f_n(x)}{2^n} \leq \frac{1}{2^n}$. And thus

$$0 \leq \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

In fact, these properties also suffice for the family of functions $f_n/2^n$ to pass the Weierstrass M-test. So the family of functions $f_n/2^n$ is uniformly convergent. And so, since each $f_n/2^n$ is continuous (the product of a continuous map with a constant map), f is continuous.

By construction, for any f_n, $f_n(X \setminus W) = \{0\}$, so $f(X \setminus W) = \{0\}$. If $x \in W$, since $W = \bigcup_{n\in\mathbb{N}} \overline{U_n}$, $x \in \overline{U_n}$ for some $n \in \mathbb{N}$. By definition, $f_n(\overline{U_n}) = \{1\}$, so $f_n(x) = 1$. So, since for every $m \in \mathbb{N}$, f_m is nonnegative, $f(x) \geq f_n(x)/2^n > 0$. So $0 \notin f(W)$, as wanted.

This finishes the proof of the claim.

Back to the main line of argument in the proof, we are now ready to the define our embedding, which we will turn later into a homeomorphism. Recall that $\mathcal{B}$ is a $\sigma-$locally finite basis for our topological space $(X, \mathcal{T})$. So $\mathcal{B}$ is the countable union of locally finite collections $\mathcal{B}_n$. We can assume without loss of generality that the $\mathcal{B}_n$ are disjoint. We could have removed any B in multiple $\mathcal{B}_n$ from all of them except one at the beginning of our proof and we would still have locally finite collections whose union was a basis.

Since the collections $\mathcal{B}_n$ are disjoint, any element $B \in \mathcal{B}$ belongs to $\mathcal{B}_n$ for a unique $n \in \mathbb{N}$. Since any such B is a basic open set, it is open and by the claim above (composing the map from the claim with a continuous scaling map) there exists a continuous map $f_B : X \to [0, 1/n]$ satisfying $0 \notin f(B)$ and $f(X \setminus B) = \{0\}$. Define our embedding $E : X \to [0,1]^{\mathcal{B}}$ by

$$E(x) := (f_B(x))_{B\in\mathcal{B}}$$

First, note that E is injective. Suppose x, y are distinct points in X. Since $(X, \mathcal{T})$ is Hausdorff, there exist disjoint open sets U, V with $x \in U$ and $y \in V$. Since $\mathcal{B}$ is a basis for $(X, \mathcal{T})$, U is the union of sets in the basis. In particular, one of the sets in the unions contains x. We have $x \in B \subseteq U$ for some $B \in \mathcal{B}$. Since $y \in V$ and V is disjoint from U, a superset of B, and by the definition of f_B, $f_B(y) = 0$ and $f_B(x) \neq 0$. So $g(x) \neq g(y)$.

Now, let's show that E is continuous. Take any open set $V \in [0, 1]^{\mathcal{B}}$ with the metric topology we described. We want to show that $g^{-1}(V)$ is open. It suffices to show that $g^{-1}(V)$ is locally open, i.e. for any $x \in g^{-1}(V)$, there is an open set U such that $x \in U \subseteq g^{-1}(V)$. Let $x \in g^{-1}(V)$. Since V is open, it is the union of open d-balls. One of these balls contains $g(x)$. So, using the triangle inequality property of d, there is an open ball $B^d_\varepsilon(g(x))$ such that $g(x) \in B^d_\varepsilon(g(x)) \subseteq V$. So $x \in g^{-1}(B^d_\varepsilon(g(x))) \subseteq g^{-1}(V)$. So it suffices to show that there is an open set U such that $x \in U \subseteq g^{-1}(B^d_\varepsilon(g(x)))$ to show that $g^{-1}(V)$ is locally open at x. This is what we will do next.

Take some $\mathcal{B}_n$. Since it is locally finite, there is an open neighborhood W_n of x that meets only finitely many of the sets in $\mathcal{B}_n$. Let $B \in \mathcal{B}_n$. If W_n does not meet B, note that $f_B(W_n) = \{0\}$, so for any $y \in W_n$, $|f_B(x) - f_B(y)| = |0 - 0| = 0$. If W_n does meet B, consider the set $O_B := (f_B(x) - \varepsilon/2, f_B(x) + \varepsilon/2) \cap [0, 1/n]$. Since it is open and f_B is continuous, $f^{-1}(O_n)$ is open. And for $y \in f^{-1}(O_n)$, $f_B(y) \in O_n$, so $|f_B(x) - f_B(y)| < \varepsilon/2$. Define U_n to be the intersection of W_n with all the sets O_B for B that intersect W_n. U_n is open because it is the finite intersection of open sets. And by the two properties just discussed, if $y \in U_n$, $|f_B(x) - f_B(y)| < \varepsilon/2$ for all $B \in \mathcal{B}_n$.

Pick $N \in \mathbb{N}$ such that $1 < \varepsilon/2 \times N$. Define $U := U_1 \cap \ldots U_N$. U is open because it is the finite intersection of open sets. Let $y \in U$. If $n \leq N$, for all $B \in B_n$, $|f_B(x) - f_B(y)| < \varepsilon/2$. If $n > N$, for all for all $B \in B_n$, $|f_B(x) - f_B(y)| < 1/n$ because the codomain of f_B is $[0, 1/n]$. And we know that $1/n < 1/N < \varepsilon/2$.

So if $y \in U$, $|f_B(x) - f_B(y)| < \varepsilon/2$ for all $B \in \mathcal{B}$. So $d(g(x), g(y)) = \sup_{B\in\mathcal{B}}\{|f_B(x) - f_B(y)|\} \leq \varepsilon/2 < \varepsilon$. So $g(y) \in B^d_\varepsilon(g(x))$. And $U \subseteq g^{-1}(B^d_\varepsilon(g(x)))$, as wanted.

Since E is continuous and injective, the map H we get by restricting E's codomain to its image is continuous, injective and surjective. Now, all that is left is to show that H is open. Let U be an open set in $(X, \mathcal{T})$. We want to show that $H(U)$ is open. It suffices to show that $H(U)$ is locally open. Let $z \in H(U)$. Since H is injective, there is a unique $x \in X$ such that $H(x) = z$. And $x \in U$. Since U is open it is the union of sets in the basis $\mathcal{B}$, one of which contains x. Call this set A. By definition of f_A, $f_A(x) \neq 0$. Define $V := \{x \in [0,1]^{\mathcal{B}} \mid x_A \neq 0\}$. Let's show that V is open by showing that it is locally open. Take $y \in V$. Consider $B^d_{y_A}(y)$. Let $w \in B^d_{y_A}(y)$. Then $d(y, w) = \sup_{B\in\mathcal{B}}\{|y_B - w_B|\} < y_A$. So $|y_B - w_B| < y_A$ for all $B \in \mathcal{B}$. And in particular $|y_A - w_A| < y_A$, so $w_A \neq 0$. So $w \in V$. So for any $y \in V$, $y \in B^d_{y_A}(y) \subseteq V$ and V is open.

Define $W := V \cap H(X)$. W is open in $H(X)$. Since $f_A(x) \neq 0$, $H(x) = z \in V$. And trivially $H(x) = z \in H(X)$. So $H(x) \in W$. Finally, let's show that $W \subseteq H(U)$. Take $t \in W$. So $t \in H(X)$ and for some $y \in X$, $H(y) = t$. And $t \in V$, so $(H(y))_A = f_A(y) \neq 0$. So by the definition of f_A, $y \in A \subseteq U$. And $H(y) = t \in H(U)$. So W is an open neighborhood of z contained in $H(U)$. So $H(U)$ is locally open and thus open, as wanted.

Since subspaces of metrizable spaces are metrizable, H is a homeomorphism from X to a metrizable space. This completes the proof of the right-to-left direction of the Nagata-Smirnov metrization theorem. ☐

Now, we can move on to the left-to-right direction of the Nagata-Smirnov theorem. By lemma 1 and lemma 2, we know that any metrizable space is Hausdorff and regular. So we just need to show that any metrizable space has a sigma-locally finite basis. We will derive this as a corollary from another strong result.

Definition 13 (Refinement of a cover)
*Let $\mathcal{C}$ be a cover of $(X, \mathcal{T})$. A **refinement** $\mathcal{D}$ of $\mathcal{C}$ is a cover of $(X, \mathcal{T})$ such that any $D \in \mathcal{D}$ is contained in some $C \in \mathcal{C}$.*

Proposition 20
Any open cover of a metrizable space has an open σ-locally finite refinement.

Proof.

Let $(X, \mathcal{T})$ be a metrizable topological space. Let d be a metric that induces $\mathcal{T}$. Let $\mathcal{C}$ be an open cover of $(X, \mathcal{T})$. By the well-ordering theorem, there is a well-order $\prec$ on the elements of $\mathcal{C}$. Define the following for any $n \in \mathbb{N}$ and any open set $U \in \mathcal{C}$:

$$S_n(U) := \{x \mid B^d_{1/n}(x) \subseteq U\}$$

$$T_n(U) := S_n(U) \setminus \bigcup_{\substack{V\in\mathcal{C},\\ V\prec U}} V$$

Claim 1: Suppose $V, W \in \mathcal{C}$ and $V \neq W$. For any $n \in \mathbb{N}$, take $x \in T_n(V)$ and $y \in T_n(W)$, then $d(x, y) \geq 1/n$.

Proof of claim 1:
Without loss of generality, assume that $V \prec W$. Since $x \in T_n(V)$, so $x \in S_n(V)$, so $B^d_{1/n}(x) \subseteq V$. And $y \in T_n(W)$ and $V \prec W$, so $y \notin V$. Suppose towards contradiction that $d(x, y) < 1/n$. Then $y \in B^d_{1/n}(x) \subseteq V$, so $y \in V$, what is a contradiction. So it must be the case that $d(x, y) \geq 1/n$.

This finishes the proof of claim 1.

For any $n \in \mathbb{N}$ and any open set $U \in \mathcal{C}$, define $E_n(U)$ as the union of open balls $B^d_{1/(3n)}(x)$ for any $x \in T_n(U)$. Note that $E_n(U)$ is open because it is the union of open sets.

Claim 2: Suppose that $V, W \in \mathcal{C}$ and $V \neq W$. For any $n \in \mathbb{N}$, take $x \in E_n(V)$ and $y \in E_n(W)$. Then, $d(x, y) > 1/(3n)$.

Proof of claim 2:
By definition of $E_n(V)$ and $E_n(W)$, there exist $w \in T_n(V)$ and $z \in T_n(W)$ such that $d(x, w) < 1/(3n)$ and $d(y, z) < 1/(3n)$. Using the triangle inequality twice,

$$d(w, z) \leq d(w, x) + d(x, z) \leq d(w, x) + d(x, y) + d(y, z)$$

By claim 1, $d(w, z) \geq 1/n$. So

$$d(x, y) \geq d(w, z) - d(w, x) - d(y, z) \geq 1/n - 1/(3n) - 1/(3n) = 1/(3n)$$

This finishes the proof of claim 2.

Claim 3: For every $U \in \mathcal{C}$ and any $n \in \mathbb{N}$, $E_n(U) \subseteq U$.

Proof of claim 3:
Let $y \in E_n(U)$. Then $y \in B^d_{1/(3n)}(x)$ for some $x \in T_n(U)$. So $x \in S_n(U)$. So $B^d_{1/n}(x) \subseteq U$. And since $y \in B^d_{1/(3n)}(x)$, $d(x, y) < 1/(3n) < 1/n$. So $y \in B^d_{1/n}(x) \subseteq U$.

This finishes the proof of claim 3.

Now we are ready to define the $\sigma-$locally finite refinement of $\mathcal{C}$. For any $n \in \mathbb{N}$, define

$$\mathcal{E}_n := \{E_n(U) \mid U \in \mathcal{C}\}$$

I claim that $\mathcal{E} := \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$ is an open $\sigma-$locally finite refinement of $\mathcal{C}$. First, recall that any $E_n(U)$ is open, as mentioned above. So this is an open collection. Next, notice that every set in $\mathcal{E}_n$ is a subset of an element of $\mathcal{C}$ by claim 3. So every set in $\mathcal{E}$ is a subset of an element in $\mathcal{C}$. Now, we just need to show that $\mathcal{E}$ is $\sigma-$locally finite and covers X. Let's first show that it is $\sigma-$locally finite.

It suffices to show that $\mathcal{E}_n$ is locally finite and this is what we will do. Take $x \in X$. I claim that $B^d_{1/(6n)}(x)$ is a neighborhood of x that intersect only finitely many sets in $\mathcal{E}_n$. in particular, it intersects at most one set in $\mathcal{E}_n$. Suppose there exists $U \in \mathcal{C}$ such that $B^d_{1/(6n)}(x) \subseteq E_n(U) \neq \varnothing$. Take any $V \in \mathcal{C}$ different from U and take $z \in E_n(V)$. By claim 2, $d(y, z) > 1/(3n)$. And by the triangle inequality and the symmetry of d, $d(x, y) + d(x, z) \geq d(y, z)$. So

$$d(x, z) \geq d(y, z) - d(x, y) > \frac{1}{3n} - \frac{1}{6n} = \frac{1}{6n}$$

So $z \notin B^d_{1/(6n)}(x)$.

Now, let's show that $\mathcal{E}$ covers X. Take $x \in X$. Since $\mathcal{C}$ covers X, some $U \in \mathcal{C}$ contains x. So, because $\prec$ is a well-order, there is a least element V of $\mathcal{C}$ that contains x. V is open, and thus the union of open balls. One of these open balls contains x and so, using the triangle inequality we get $x \in B^d_\varepsilon(x) \subseteq V$ for some $\varepsilon > 0$. Pick N such that $1 < \varepsilon \times N$. Then $x \in B^d_{1/N}(x) \subseteq V$. So $x \in S_N(V)$. And because V is the least element containing x, $T_N(V) = S_N(V)$. So, by the definition of $E_N(V)$, $B^d_{1/(3N)}(x) \subseteq E_N(V)$. So $x \in E_N(V) \subseteq \mathcal{E}_N \subseteq \mathcal{E}$. $\square$

Corollary 21

Any metrizable space has a σ-locally finite basis.

Proof.

Let $(X, \mathcal{T})$ be a metrizable space. Let d be a metric that induces $\mathcal{T}$. For each $n \in \mathbb{N}$, define $\mathcal{C}_n := \{B^d_{1/n}(x) \mid x \in X\}$. Since for any $x \in X$, $x \in B^d_{1/n}(x)$, $\mathcal{C}_n$ is an open cover of X. By the proposition above, for each $n \in \mathbb{N}$, there is an open $\sigma-$locally finite refinement $\mathcal{D}_n$ of $\mathcal{C}_n$.

Claim: Suppose $D \in \mathcal{D}_n$ and $a, b \in D$. Then $d(a, b) < 2/n$.

Proof of claim:
Because $\mathcal{D}_n$ is a refinement of $\mathcal{C}_n$, $D \subseteq B^d_{1/n}(x)$ for some $x \in X$. So $a, b \in B^d_{1/n}(x)$. In other words, $d(a, x) < 1/n$ and $d(x, b) < 1/n$. And by the triangle inequality, $d(a, b) \leq d(a, x) + d(x, b) < 1/n + 1/n = 2/n$.

This finishes the proof of the claim.

Back to the main line of argument in this proof, define $\mathcal{D} := \bigcup_{n \in \mathbb{N}} \mathcal{D}_n$. $\mathcal{D}$ is the countable union of countable unions of locally finite sets, so it is the countable union of locally finite sets, i.e. $\sigma-$locally finite. Now, all that is left to do is show that $\mathcal{D}$ is a basis. We will show that any open set is the union of sets in $\mathcal{D}$. Take any open set $U \subseteq X$. Take any $x \in U$. Since U is open, it is the union of open balls, one of which contains x. So using, the triangle inequality, we get $x \in B^d_\varepsilon(x) \subseteq U$ for some $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that $2 < \varepsilon \times N$. $\mathcal{D}_N$ covers X, so there is some $D \in \mathcal{D}_N \subseteq \mathcal{D}$ such that $x \in D$. By the claim above, for any $y \in D$, $d(x, y) < 2/N < \varepsilon$. So $y \in B^d_\varepsilon(x)$. So $x \in D \subseteq B^d_\varepsilon(x) \subseteq U$. Since x was arbitrary, for any element x of U we have $x \in D_x \subseteq U$ for some $D_x \in \mathcal{D}$. So $U = \bigcup_{x \in X} D_x$ for $D_x \in \mathcal{D}$. $\square$

This finishes the proof of the Nagata-Smirnov metrization theorem.

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