Nets and Filters

Or: Net fixes (and chill)

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This article assumes background in introductory analysis, specifically sequences in metric spaces, and point-set topology through the first half of this course. The goal is to convince the reader that analyzing convergent sequences can fail to capture important properties of topological spaces such as closure, continuity, and compactness. The main question we will try to answer is: *how do we fix this?* We will define nets to be the analogous notion of sequences in an arbitrary topological space. We will then show that nets can be used interchangeably with filters, another generalization of sequences in a topological space.

1 Where sequences fail

1.1 Convergent sequences in metric spaces

Recall that for a set X, a sequence $\{x_n\} \subset X$ **converges** to a point $x \in X$ if for every open set containing x, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Or, more simply put, a sequence $\{x_n\}$ converges to x if every open set around x contains a tail of the sequence. In this case, we write $x_n \to x$. A point x is a **limit point** of a sequence $\{x_n\}$ if every open set U around x contains such an infinite tail of $\{x_n\}$.

Many properties of convergent sequences in the Euclidian space generalize nicely to any arbitrary metric space (X, d). For instance:

Nice property 1. *Every sequence in (X, d) converges to at most one point.*

In metric spaces, we can also use convergent sequences to characterize closure, continuous functions between metric spaces, compactness, and limit points.

Nice property 2. *There is a sequence in $A \subset X$ converging to a iff $a \in \overline{A}$.*

Nice property 3. *$f : (X, d) \to (Y, d)$ is continuous iff for every convergent sequence $x_n \to x$ in X, $f(x_n) \to f(x)$ in Y.*

Nice property 4. *(X, d) is compact iff every sequence has a convergent subsequence.*

Nice property 5. *x is a limit point of a sequence $\{x_n\}$ in (X, d) iff there is a subsequence $\{x_{n_k}\}$ of x_n converging to x.*

Looking at convergent sequences and subsequences gives us a way to study the important topological notions in metric spaces. Metric spaces are neat, but we want to be able to achieve something similar for *any* topological space.

1.2 Convergent sequences in topological spaces

The definition of a convergent sequence remains the same for any space: a sequence converges to a point if we can fit a tail of the sequence in any open set around that point. Note that we can also say "all but finitely many points" of the sequence are in every open set around the point.

Example 1. In a discrete space, only sequences with constant tails (terms take on a constant value far enough along in the sequence) converge. Since singletons are open sets, in order for the tail of the sequence to be in the open set around a point, the tail must precisely be that point.

Example 2. In an indiscrete space, *every* sequence converges to *every* point! This is because an open set around one point contains every other point. In $(\mathbb{R}, \mathcal{T}_{\text{indisc}})$, then, pick any sequence, and it converges to 0. Or 1,000,000. Or -333. Or π. You get the idea.

These examples give us the sense that sequences in non-metric spaces can behave weirdly. Oftentimes, we throw conditions at statements to prevent weird behavior. Indeed, we will need additional constraints in order to replicate the nice properties above for any topological space. From now on, we will use X to indicate a topological space.

Nice-ish property 1. *If X is a Hausdorff space, then every sequence in X converges to at most one point.*

Proof. Suppose $x_n \to x$. Take $y \in X$, $y \neq x$. Since X is Hausdorff, there exist disjoint, open sets U, V such that $x \in U, y \in V$. But if a tail of $\{x_n\}$ is contained in U, then only finitely many points of $\{x_n\}$ are in V. So $x_n \not\to y$. $\square$

Since metric spaces are Hausdorff, this is a more general result than Nice property 1. As in the case of metric spaces, we want to say that the converse is true, but unfortunately, it is not. Here is a counterexample:

Example 3. Let X an uncountable set with the co-countable topology, such as $\mathbb{R}_{cc}$. A sequence $\{x_n\}$ in X converges to x iff $\{x_n\}$ has a constant tail with value x, i.e. $x_n = x$ for large enough n. To see this, suppose $x_n \to x$. Let

$$A = \{x_n \mid x_n \neq x\}.$$

A is countable, so $X \setminus A$ is uncountable, which means that A is closed. Since $x \notin A$, $X \setminus A$ is an open set containing x. Thus a tail of $\{x_n\}$ must be in $X \setminus A$.

But $\{x_n\} \cup X \setminus A = \{x\}$, so the tail is constant. Conversely, suppose that $x_n = x$ for $n \geq N$, $N \in \mathbb{N}$. Let U be an open set containing x. Then for all $n \geq N$, $x_n \in U$. Since U is arbitrary, $x_n \to x$ by definition of convergence. So every sequence in this space converges to at most one point. However, we know that the co-countable space is not Hausdorff.

With this example, we can already see that we cannot rely solely on analyzing convergent sequences to reveal the underlying data of a topology. Though $\mathbb{R}_{cc}$ is very different from the discrete space, their convergent sequences act the same. This feels troubling. In particular, notice in the above example that to prove a sequence with a constant tail of x converges to x, we didn't use any properties of the co-countable topology. So this statement holds for any topological space.

Anyway, the Hausdorff condition is important here, because it guarantees that we can separate points by open sets. Remember that we determine convergence by examining *all* of the open sets around a point. If two points are *topologically indistinguishable*, i.e. there are no open sets seperating them, then it makes sense that a sequence can converge to both points simultaneously.

In the extreme example of the indiscrete space, *all* points are topologically indistinguishable. This presents the awkward case we saw above, where every sequence converges to every point.

For our other nice properties to hold, we will need conditions beyond Hausdorffness.

Definition 1.1. *$A \subset X$ is **sequentially open** if for a sequence $\{x_n\} \in X$ that converges to $x \in A$, the tail of $\{x_n\}$ is contained in A. $S \subset X$ is **sequentially closed** if for a sequence $\{x_n\} \subset S$ that converges to $x \in X$, $x \in S$.*

It is equivalent to say that A is sequentially open iff no sequence in $X \setminus A$ converges to a point in A.

Proposition 1.1. *For a metric space X, $A \subset X$ is open iff A is sequentially open.*

Proof. If $A \subset X$ is open, $\{x_n\}$ a sequence in $X \setminus A$, and $a \in A$, then A is an open set containing a but not any element of $\{x_n\}$. So $\{x_n\}$ cannot converge to a, and A is sequentially open.

Conversely, if $A \subset X$ is not open, then any open set around some $a \in A$ has non-trivial intersection with $X \setminus A$. Pick

$$x_n \in (X \setminus A) \cap B_{\frac{1}{n+1}}(a)$$

for all $n \in N$. Then $\{x_n\}$ is in $X \setminus A$ but converges to $a \in A$. Thus A is not sequentially open. $\square$

Hence the two notions are the same in a metric space. Notice that for the first implication, we did not use the fact that X is a metric space, so the following statement is true for any topological space.

Proposition 1.2. *For any space X, if $A \subset X$ is open, then A is sequentially open.*

This proposition also says that a closed subset of X is also sequentially closed. As we just saw for metric spaces, the reverse is also true.

X is **sequential** if this is the case, i.e. if sequentially open sets are open and, equivalently, if sequentially closed sets are closed.

Following the theme of much of this section, however, we must once again hesitate before believing the reverse to be true for *all* spaces.

Example 4. Once again, let X be a set with the co-countable topology. Let $\{x_n\}$ be a sequence in $S \subset X$ converging to some $x \in X$. Let

$$M = \{x_n | x_n \neq x\}.$$

M is countable, so M is closed in X, which means that $X \setminus M$ is open and contains x. Since $x_n \to x$, $X \setminus M$ contains a tail of $\{x_n\}$. Thus there must be $N \in \mathbb{N}$ large enough such that $x_N = x$. Then $\{x_n\} \subset S$ gives us that $x \in S$ and every subset S is sequentially closed. However, we know that every uncountable subset in X is not closed. So X is not sequential.

We thus have the fact:

Proposition 1.3. *Not all topological spaces are sequential.*

When dealing with a space that is not sequential, it becomes harder to use the notions of sequentially open and sequentially closed, and therefore sequences, to exactly define the topology. In a metric space, we often think of convergent sequences as "getting close to something." Even if that "something" is not in the set containing the sequence, our sequence still approach it.

Sequential spaces end up being the ones where we can "do business as usual" with sequences to determine the important properties of the topology. We can think of the open sets in a sequential space as "mimicking" the structure of a metric space.

So we can transfer over what we know about metric spaces to sequential spaces, but there is in fact a *stronger* property that allows us to use convergent sequences to extract information from a topology.

Definition 1.2. *A **countable basis** at a point $x \in X$ is a countable collection of open sets $\mathcal{B}_x$ around x such that for every open set U containing x, there is some $B \in \mathcal{B}_x$ with $B \subset U$.*

*A space X is **first countable** if every point $x \in X$ has a countable basis.*

Proposition 1.4. *Every metric space is first countable.*

Proof. The set of open balls around x with radius $\frac{1}{n}$ for $n = 1, 2, 3, ...$ form a countable basis at x. $\square$

We have analogues of our nice properties for first countable spaces.

Nice-ish property 2. *If X is first countable, there is a sequence in $A \subset X$ converging to a iff $a \in \overline{A}$.*

Nice-ish property 3. *If X is first countable, then $f : X \to Y$ is continuous iff for every convergent sequence $x_n \to x$ in X, $f(x_n) \to f(x)$ in Y.*

Nice-ish property 5. *If X is first countable, x is a limit point of a sequence $\{x_n\}$ in X iff there is a subsequence $\{x_{n_k}\}$ of x_n converging to x.*

Proof. Complete proofs for the three statements above are available in Reference 3. $\square$

But why is this first countability condition important? The underlying idea is that if there is a countable basis at a point x in X, then there is a countable *and nested* basis of open sets around x,

$$B_1 \supset B_2 \supset ...$$

such that every open set U around x contains some B_i. You can prove this by noticing that the finite intersection of basis elements is also a basis element.

Indeed, in metric spaces, the set of open $\frac{1}{n}$-balls around x form a *nested* countable basis at x:

$$B_1(x) \supset B_{\frac{1}{2}}(x) \supset B_{\frac{1}{3}}(x)...$$

In a non-metric space, we can imagine this sequence of nested basis elements taking on the same role as the open $\frac{1}{n}$-balls. Since sequences are countable, the fact that there are only countably many of these nested basis elements ensures that terms of a sequence can approach a point x.

If every basis element B_i around x contains a term of the sequence, then because the B_i's are nested and any open set around x contains some B_i, *every* open set around x will contain a tail of the sequence.

Without a countable basis around a point, we don't know if a sequence will ever get "close enough" to a point such that every open set around that point will contain its tail.

Example 5. An uncountable set X with the co-countable topology is not first countable. To see this, let's assume that X is first countable. Let $x \in X$. Then there exists a countable basis $\mathcal{B}_x$ around x. For any $B \in \mathcal{B}_x$, B is open by definition of a basis, so $X \setminus B$ is countable. Then

$$\bigcup_{B \in \mathcal{B}_x} (X \setminus B)$$

is a countable union of countable sets, thus countable. But

$$\bigcup_{B \in \mathcal{B}_x} (X \setminus B) = X \setminus \bigcap_{B \in \mathcal{B}_x} B = X \setminus \{x\}$$

gives us that $X \setminus \{x\}$ is countable. However, since X is uncountable, it remains uncountable after deleting a point, so we reach a contradiction.

Because $\mathbb{R}_{cc}$ is not first countable, sequences can't exactly see what its open sets look like, leading them to believe that $\mathbb{R}_{cc}$ is discrete!

Having the property of first countability is sufficient to talk about sequences in such spaces. But can we do better? While many spaces are first countable, we want something that can encapsulate the notion of convergent sequences for *all* topological spaces. To do so, we'll have to think beyond sequences. And this is where nets come in.

2 Nets

Per our discussion above, we can think of sequences as being too short to probe uncountable sets. Sequences are indexed by the natural numbers, thus countable. We want something "longer." We also concluded that sequences can fail to retrieve all the data of a topological space. In this sense, the view from above that sequences provide is too narrow. Sequences can only approach a point from one direction, whereas we want to be able to approach a point from *all* directions. This will give us a more transparent picture of what is really going on in a topological space.

By using a directed set in place of $\mathbb{N}$ as our indexing set, nets will fix these grievances we have with sequences.

Definition 2.1. *Let D be a set and $\leq$ be a relation on D such that:*

1. *$\leq$ is reflexive: for any $x \in D$, $x \leq x$*
2. *$\leq$ is transitive: for any $x, y, z \in D$, if $x \leq y$ and $y \leq z$, then $x \leq z$*
3. *$\leq$ is directed: for any $x, y \in D$, there exists $c \in D$ such that $x \leq c$ and $y \leq c$, i.e. any two elements in D have an upper bound*

Then D with the relation $\leq$ is a ***directed set****.*

Example 6. Some examples of directed sets:

1. $\mathbb{N}$ with the usual ordering $\leq$ ("less than or equal to") is a directed set.
2. If $(X, \mathcal{T})$ is a topological space and $x \in X$, then the set $\{U \in \mathcal{T} \mid x \in U\}$ with the subset relation $\subseteq$ is a directed set. $\subseteq$ clearly satisfies the first two conditions above. To see that it satisfies the third, notice that if U, V are open sets containing x, then $U \cup V$ is an open set containing x, and $U, V \subseteq U \cup V$. The same set equipped with the relation $\supseteq$ is also a directed set.

3. If D, E are directed sets, then $D \times E$ is a directed set by defining the relation $(d_1, e_1) \leq (d_2, e_2)$ iff $d_1 \leq d_2$ in D and $e_1 \leq e_2$ in E.

   For instance, although $\mathbb{N} \times \mathbb{N}$ with the usual ordering on each component is not a totally ordered set ($(0,1)$ and $(1,0)$ are incomparable), it is a directed set. Any $(n_1, n_2), (m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ is less than or equal to $(\max\{n_1, m_1\}, \max\{n_2, m_2\})$.

Definition 2.2. *A* ***net*** *in a topological space X is a function $w : D \to X$, where D is a directed set.*

Note that a sequence is thus a net $w : \mathbb{N} \to X$, where $\mathbb{N}$ has the usual ordering $\leq$.

A net **converges** to a point $x \in X$ if for every open set U containing x, there exists $d \in D$ such that for all $e \geq d$, $w(e) \in U$. Similar to how we defined convergent sequences, w converges iff every open set around x contains a tail of w, where the tail is defined as

$$T(e) = \{w(e) \mid e \geq d \in D\}.$$

We write $w \to x$ and call x a **limit point** of the net w.

When $D = \mathbb{N}$, this becomes precisely the definition of a convergent sequence. The net w simply maps a natural number to the element of the sequence it indexes. For large enough $n \in \mathbb{N}$, $w(n) = x_n$ is in every open set around its limit point.

We saw that when we impose the criterion of Hausdorff-ness, sequences have unique limit poins. However, as shown by the co-countable topology example, the converse is not necessarily true. Now that we have nets, we can completely characterize Hausdorff spaces using convergent nets.

Net property 1. *X is a Hausdorff space iff every net in X converges to at most one point.*

Proof. Let X be a Hausdorff space and w be a net in X. Suppose that $w \to x$. Take $y \in X, y \neq x$. We want to show that $w \not\to y$. Since X is Hausdorff, we can find disjoint open sets U, V such that $x \in U$, $y \in V$. By convergence of w, a tail of w,

$$T(e) = \{w(e) \mid e \geq d \in D\}$$

is contained in U and disjoint from V. If $w \to y$, then there is a tail of w,

$$T(f) = \{w(f) \mid f \geq d\}$$

in V and disjoint from U. Since D is a directed set, there exists $c \in D$ such that $e \leq c$ and $f \leq c$. Then

$$T(c) = \{w(c) \mid c \geq e, c \geq f\}$$

is a tail of w in both U and V, which cannot happen since U, V are disjoint. Thus w cannot converge to y.

Conversely, suppose X is not Hausdorff. We want to show that there is a net w converging to more than one point. Take two points $x, y \in X, x \neq y$ such that any two open sets U, V containing x and y, respectively, have nontrivial intersection. Let $D = D_x \times D_y$, where

$$D_x = \{U \in \mathcal{T} \mid x \in U\} \text{ and } D_y = \{V \in \mathcal{T} \mid y \in V\}$$

and the relation is defined

$$(U_1, V_1) \leq (U_2, V_2) \text{ iff } U_1 \supseteq U_2 \text{ and } V_1 \supseteq V_2.$$

For each $(U, V) \in D$, take any $x_{U,V} \in U \cap V$. Define a net $w : D \to X$ as $w((U, V)) = x_{U,V}$. Then w converges to both x and y.

To see this, take an open set U_0 containing x. Then $U_0 \in D_x$, and $(U_0, X) \in D$. Note that X is itself an open set containing y. Take any $(U, V) \in D$ such that $(U_0, X) \leq (U, V)$. Certainly $X \supseteq V$, so this specifically tells us that $U_0 \supseteq U$. Then

$$x_{U,V} \in U \cap V \subseteq U_0 \cap V \subseteq U_0.$$

In other words, for an open set U_0 containing x, there exists $(U_0, X) \in D$ such that for all $(U, V) \geq (U_0, X)$, $w((U, V)) = x_{U,V} \in U_0$. Therefore $w \to x$. Taking an open set V_0 containing y, we can similarly show that $w \to y$. $\square$

We said earlier that when a space is not first countable, the view from above that sequences provide is too narrow. In the proof above, we see that nets can in fact traverse through *all* the open sets containing a point, thereby patching up the gaps that ordinary sequences might encounter.

You may, at this point, feel slightly uneasy taking arbitrary elements, such as $x_{U,V} \in U \cap V$ above. We quickly note that we can quell these set-theoretic worries, here and in later proofs, by invoking the axiom of choice.

Net property 2. *There is a net $w : D \to A$ in $A \subset X$ converging to a iff $a \in \overline{A}$.*

Proof. Suppose $w : D \to A$ is a net in A converging to a. By definition of convergence, for an open set U containing a, there exists $d \in D$ such that $w(e) \in U$ for all $e \geq d$. Recall that $a \in \overline{A}$ iff for all open sets containing a, $U \cap A$ is non-empty. But by definition of a net, $w(e) \in A$ for all $e \in D$. So $U \cap A$ is non-empty.

Conversely, suppose $a \in \overline{A}$. Then for every open set U containing a, there exists some $a_U \in U \cap A$. Let

$$D_a = \{U \in \mathcal{T} \mid a \in U\}$$

be a directed set with the relation $\supseteq$, and define the net $w : D_a \to A$ by $w(U) = a_U$. We want to show that the tail of w is contained in U, and thus $w \to a$. For $U \in D_a$ and $V \in D_a$ with $U \leq V$, i.e. $U \supseteq V$, we have

$$w(V) = a_V \in V \cap A \subseteq U \cap A \subseteq U.$$

So for $V \geq U$, $w(V) \in U$, which is what we wanted to show. $\square$

We used, in these two proofs, the directed set consisting of open sets around x with the superset relation $\supseteq$. Intuitively, by going further along this directed set, we can get "close" to a point by narrowing our field of vision to smaller and smaller open sets. In both proofs, our net "leapfrogs" through these open sets by continually jumping to a point in the intersections. We will continue to use this particular directed set along with the "leapfrogging" method in later proofs.

In addition to closure, convergent nets can completely characterize continuous functions in a topological space.

Net property 3. *For topological spaces X, Y, $f : X \to Y$ is continuous iff for every net $w : D \to X$, $w \to x$ in X, the net $f(w) \to f(x)$.*

Proof. Suppose f is continuous and $w \to x$ in X. Let U_Y be an open set in Y containing $f(x)$. Then by continuity of f, $f^{-1}(U_Y)$ is an open set in X containing x. $w \to x$ means that there exists $d \in D$ such that $w(e) \in f^{-1}(U_Y)$ for all $e \geq d$. Then $f(w(e)) \in U_Y$ for all $e \geq d$ which implies that $f(w)$ converges to $f(x)$ in Y.

Conversely, suppose that f is not continuous. Then there is some open set $U_Y \subset Y$ such that $f^{-1}(U_Y) \subset X$ is not open. We know that a set is open if and only if it is locally open. Here, $f^{-1}(U_Y)$ is not open implies that $f^{-1}(U_Y)$ is not locally open. By definition of locally open, then, there exists $x \in f^{-1}(U_Y)$ such that every open set U around x has a point $x_V \notin f^{-1}(U_Y)$. Once again, consider the directed set

$$D = \{U \in \mathcal{T} \mid x \in U\}$$

with the relation $\supseteq$, and let $w : D \to X$ be a net defined by $w(V) = x_V$ for $V \in D$. Then w converges to x, since for $V \geq U$, i.e. $V \subseteq U$, we have

$$w(V) = x_V \in V \subseteq U.$$

Now suppose the net $f(w)$ converges to $f(x)$ in Y. Since $U_Y \subset Y$ is an open set containing $f(x)$, there is some $V_Y \in D$ such that $f(x_V) \in U_Y$ for all $V \geq V_Y$. But this implies $x_V \in f^{-1}(U_Y)$, contradiction. So $f(w)$ cannot converge to $f(x)$. $\square$

Like how we equate continuity with sequential continuity in metric spaces, we can continue to think of continuity in general topological spaces in terms of "net-sequential" continuity.

2.1 Subnets

We will need some analogue of a subsequence to state the last of the nice net properties.

(Tentative) Net property 4. *X is compact iff every net has a convergent subnet.*

(Tentative) Net property 5. *x is a limit point of a net $w : D \to X$ iff there is a subnet of w converging to x.*

Before we can set these properties into stone, we need to determine what the net-analogue of a subsequence is. Our goal is to define the notion of a subnet such that these properties are true.

Definition 2.3. *A subset D' of a directed set D is* ***cofinal*** *if for every $d \in D$, there exists an element $e \in D'$ such that $d \leq e$.*

A cofinal subset of a directed set is also directed.

Example 7. Some examples of cofinal sets:

1. A subset of $\mathbb{N}$ is cofinal iff it is infinite.
2. The diagonal $\triangle_{\mathbb{N}} = \{(n, n) \mid n \in \mathbb{N}\}$ is a cofinal subset of $\mathbb{N} \times \mathbb{N}$. Any $(n, m) \in \mathbb{N} \times \mathbb{N}$ is less than or equal to $(\max\{n, m\}, \max\{n, m\}) \in \triangle_{\mathbb{N}}$.
3. Take $D_x = \{U \in \mathcal{T} \mid x \in U\}$ with the superset relation, i.e. $U \leq V$ iff $U \supseteq V$, the directed set we know and love. A cofinal subset D'_x of D_x requires that for every $U \in D_x$, there exists $B \in D'_x \subseteq D_x$ such that $B \geq U$, i.e. $B \subseteq U$. Look familiar? The cofinal subset of open sets around x precisely forms a local basis around x!

So cofinal subsets are a sort of "unbounded" subset in D where we can find an element "above" any element in D. Since we define a subsequence as an infinite subset of a sequence indexed by an "unbounded" subset of $\mathbb{N}$, you might think (and secretly hope) that the definition of a subnet goes something like this:

A subnet of $w : D \to X$, say $v : E \to X$, is a net where E is a cofinal subset of D.

However, this definition of a subnet does not quite give us the flexibility we want to prove the net properties above. (See addendum for an example.)

Instead, we define a subnet as follows:

Definition 2.4. *Let D, E be directed sets and $w : D \to X$, $v : E \to X$ be nets. v is a* ***subnet*** *of w if there is a function $f : E \to D$ such that:*

1. *f is monotone: if $e_1 \leq e_2$, then $f(e_1) \leq f(e_2)$.*
2. *f is cofinal: $f(E)$ is a cofinal subset of D, i.e. for all $d \in D$, there exists $e \in E$ such that $d \leq f(e)$.*

3. *v factors through D: $v(e) = w(f(e))$ for all $e \in E$.*

$$\begin{tikzcd}
E \arrow[r, "v"] \arrow[dr, "f"'] & X \\
 & D \arrow[u, "w"']
\end{tikzcd}$$

Note that the function f we defined above does not need to be injective.

As an example, let's take the nets $w : D \to \mathbb{N}$ and $v : E \to \mathbb{N}$, where $D = E = \mathbb{N}$ with the usual ordering $\leq$. Define $f : E \to D$ as

$$f(0) = 0$$

$$f(e) = e - 1 \text{ for } e > 0$$

Let w be the identity map, i.e. $w(d) = d$, and let $v(e) = f(e)$. We can check that v is a subnet of w: f is indeed monotone and cofinal, and $v(e) = f(e) = w(f(e))$. f maps both 0 and 1 in E to 0 in D, so f is not injective.

Even if the directed set indexing a net is a sequence, e.g. $D = \mathbb{N}$, the directed set E indexing a subnet *does not* need to be $\mathbb{N}$ or a subset of $\mathbb{N}$. The domain of the subnet can even have a larger cardinality than the domain of the net itself!

So while subsequences are restrictions of a sequence, subnets—somewhat counterintuitively—can actually elongate a net. Therefore not every subnet of a sequence has to be a subsequence. In particular, a sequence, if regarded as a net, can have more subnets than subsequences. We will store this fact in our back pocket for now. After proving Net property 5, we will muster up some intuition behind why this should be the case.

Okay, *now* we can settle our tentative net properties, restated below.

But first, a lemma that will help us:

Lemma 2.1. *X is compact iff every set of closed subsets $\{C_i \subset X\}_{i \in I}$, in which the intersection of finitely many C_i is nonempty, has nonempty total intersection, i.e.*

$$\bigcap_{i \in I} C_i \neq \emptyset.$$

Proof. Proof in Munkres (Theorem 26.9). □

Net property 4. *X is compact iff every net has a convergent subnet.*

Proof. We will provide a detailed sketch here. The full proof can be found in Reference 3.

Let's first assume X is compact. Let $w : D \to X$ be a net. We want to find a subnet $v : E \to X$ that converges. To do this, we look at the tails of w,

$$T(e) = \{w(e) \mid e \geq d \in D\}.$$

By directedness of D, the intersection of finitely many of these tails $T(e)$ is nonempty. Hence, the intersection of finitely many of their closures $\overline{T(e)}$ is nonempty. Thus from the lemma, there exists

$$x \in \bigcap_{e \geq d} \overline{T(e)}.$$

That means every open set U_x around x contains points in every $T(e)$ for $e \geq d$. By definition of $T(e)$, then, for every $d \in D$, there exists $e \geq d$ such that $w(e) \in U_x$.

All of this sets up what we came here to do: construct a subnet $v : E \to X$ converging to x. I will call this next part of the proof the *"subnet cherry-picking method,"* as we will use it again later. First, we must specify a directed set E. Let $\mathcal{U}_x$ be the set of all open sets containing x. Let

$$E = \{(d, U) \in D \times \mathcal{U}_x \mid w(d) \in U\}$$

with the relation

$$(d_1, U_1) \leq (d_2, U_2) \text{ iff } d_1 \leq d_2 \text{ in } D \text{ and } U_1 \supseteq U_2.$$

(One can check that this is a directed set.) Next, we define a function $f : E \to D$ as:

$$f((d, U)) = d.$$

We use this to define a subnet $v = w \circ f$. (One can check that this is indeed a subnet). The idea here is that f "projects" onto the desirable elements of D, namely those such that $w(d)$ is in an open set containing x. More formally,

$$v((d, U)) = w(f((d, U))) = w(d) \in U$$

where U is an open set containing x. This fact, combined with how we defined x above, we can fairly quickly conclude that $v \to x$.

Conversely, let's assume that X is not compact. We want to construct a net without a convergent subnet. By the lemma, there is then a set of closed subsets of X, $\{C_i\}$ with non-empty intersection of finitely many elements, but empty total intersection. The strategy for this proof is to take D to be the finite subcollections of $\{C_i\}$ with the superset relation. Let $w : D \to X$ be the net sending such a subcollection to a point in their intersection. Using the fact that the total intersection of $\{C_i\}$ is empty, we can find a point that is not in C_i for some i. Since C_i is closed, there is an open set U containing x such that $U \cap C_i = \emptyset$. Roughly speaking, for a subnet $v : E \to X$ of w, we can use the cofinality of $f(E)$ to "bound" C_i. If we assume that v converges, then w ends up mapping to an element simultaneously in U and not in U. So v cannot converge.

We have skipped some steps toward the end, but the idea here is that, since we are concerned with convergence, we use properties of directness and cofinality to "translate" between the indexing set of a net and its subnet.

□

Recall that this property, phrased in terms of sequences and subsequences, was true for metric spaces. In other words, a metric space is compact iff it is sequentially compact. Thinking "net-sequentially," then, we can characterize compactness in an arbitrary topological space.

Net property 5. *x is a limit point of a net $w : D \to X$ iff there is a subnet of w converging to x.*

Proof. We will again provide a sketch here. The full proof can be found in Reference 2.

To prove the forward direction, notice that x as defined in the forward direction of the previous proof is in fact a limit point. So we can use the *"subnet cherry-picking method"* by taking the same subnet $v : E \to X$ of a net $w : D \to X$ and the same "projection" function. This shows that v converges to x.

Conversely, if $w : D \to X$ is a net, and $v : E \to X$ is a subnet of w, with $v \to x$, we want to show that x is a limit point of w. Let $f : E \to D$ be the function defining v as a subnet of w. Let U be an open set containing x. Since f is cofinal, for any $d \in D$, there is an $\alpha \in E$ such that $f(\alpha) \geq d$. Since $v \to x$, there is a $\beta \in E$ such that $v(\gamma) \in U$ for all $\gamma \geq \beta$. By directness of E, there is some $\delta \in E$ such that $\delta \geq \alpha$ and $\delta \geq \gamma$. By monotonicity of f, then, $f(\delta) \geq f(\alpha) \geq d$. So if we let $e = f(\delta) \in D$, then $w(e) = w(f(\delta)) = v(\delta) \in U$. Therefore $v \to x$. The idea here is that if an open set around a point contains a tail of v, then we can use properties of directedness and subnets to find a tail of w that is also in that open set. □

As promised, let's try to put together some intuition behind why we define subnets the way we do. In a metric space, if a sequence approaches a point x, we can construct a subsequence also approaching x by choosing terms of the sequence that get closer and closer to x. We can do this, because there is a countable basis of open $\frac{1}{n}$-balls around x. As long as we choose a term in each of these $\frac{1}{n}$-balls, our subsequence will converge.

But now, we don't necessarily have a countable basis at x. Remember how—in the proofs of net property 1 and 2—we constructed a convergent net by taking the open sets around x and "leapfrogging" through them to approach x? This is how we made sure that the net will eventually lie in every open set around

x. The challenge we face is that a convergent subnet has to essentially do the same thing.

So we have to ask: given a net with a limit point, how do we construct a subnet so that the subnet converges? Without a countable, nested basis at x, it seems like we don't have enough to work with if we're restricted to choosing terms from the net.

This is why, to produce convergent subnets in the proofs above, we had to take special care to "cherry-pick" not only a term of the net, but also an open set containing it, hence the *"subnet cherry-picking method."* We repeated the process by taking smaller and smaller open sets containing terms of the net so that the subnet eventually lay in every open set around x. Therefore to do our "cherry-picking," we must allow a subnet to have a larger domain than a net.

Phew! That was a lot of work, but the payoff is worth it. We've just proved that all of our nice properties of metric spaces transfer over to any topological space if we phrase them in the language of of nets and subnets.

Our last hope now is to show that nets "know everything about" the topology itself. As we saw with the example of the discrete and co-countable topologies, sequences acted "blind" towards the nature of these two topologies. Lucky for us, the all-seeing eye of convergent nets completely determines a topology.

Theorem 2.2. *Given a topological space X, $A \subset X$ is open iff no net in $X \setminus A$ has a limit point in A.*

Proof. Suppose $A \subset X$ is open, and let $w : D \to X \setminus A$ be a net in $X \setminus A$. Take $a \in A$. Since A is open, then there is an open set U around a that is contained in A. Then U does not contain any elements of w, so w cannot converge to a, i.e. a is not a limit point of w.

Conversely, suppose A is not open. Thus there exists some $a \in A$ such that every open set U around a has a point $x_U \notin A$, i.e. $x_U \in (X \setminus A) \cap U$. We want to show that there is a net in $X \setminus A$ converging to a. Let

$$D = \{U \in \mathcal{T} \mid a \in U\}$$

with the relation $\supseteq$, and let $w : D \to X \setminus A$ be a net defined by $w(U) = x_U$. Hence for $V \geq U$, i.e. $V \subseteq U$,

$$w(V) = x_V \in (X \setminus A) \cap V \subseteq (X \setminus A) \cap U.$$

So w is a net in $X \setminus A$ converging to $a \in A$. $\square$

Since a topology is defined by its open sets, and we've just shown that open sets can be defined by convergent nets, we can deduce the following:

Theorem 2.3. *Two topologies $\mathcal{T}_1, \mathcal{T}_2$ on a set X are equivalent iff every net that converges in $\mathcal{T}_1$ also converges in $\mathcal{T}_2$, and every net that converges in $\mathcal{T}_2$ also converges in $\mathcal{T}_1$.*

Therefore we can define topological spaces using *only* the convergence of nets.

3 Filters

If nets don't float your boat (or even if they do), we will briefly introduce filters as an alternative generalization of sequences in arbitrary topological spaces.

Definition 3.1. *A **filter** on a topological space X is a nonempty collection $\mathcal{F} \subseteq \mathcal{P}(X)$ such that:*

1. $\emptyset \notin \mathcal{F}$
2. *$\mathcal{F}$ is closed under supersets: if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.*
3. *$\mathcal{F}$ is closed under finite intersection: if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.*

A filter $\mathcal{F}$ on X **converges** to a point $x \in X$ if for every open set U containing x, $U \in \mathcal{F}$. In this case, we write $\mathcal{F} \to x$.

Example 8. Some examples of filters:

1. Trivially, $\mathcal{F} = \{X\}$ is a filter on X, provided that X is nonempty.
2. $\mathcal{N}_x = \{A \subseteq X \mid \exists$ an open set U with $x \in U \subseteq A\}$ is a filter on X. This is called the **neighborhood filter** of x (the set of subsets where x is an interior point).
3. $\mathcal{P}_x = \{A \subseteq X \mid x \in A\}$ is the **principal filter** on X (the set of subsets containing x).
4. A filter $\mathcal{F}$ on X is an **ultrafilter** if for any $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

In our non-mathematical lives, we would use filters (strainers, colanders, etc.) to keep the large chunks—say, oh I don't know, pasta—and get rid of the small chunks, pasta water in this case. Filters on a space X, fittingly, "filter out" the chunks of X it deems to be small while keeping the large chunks.

To filters, a subset of X is "large" if it contains something. In the principal filter, the *something* is simply the point x. In the neighborhood filter, the *something* is an open set around the point x.

We see this encoding of "largeness" from the definition. The empty set does not contain anything thus cannot be large. A set containing a large set must be large, since it will also contain the *thing* we are looking for. Finally, the intersection of large sets must also contain the *thing*, so the intersection is large as well.

Ultrafilters can do even better, since for every subset of X, they can tag either the subset or its complement as large.

This is all we need to know about filters to say that...

4 Equivalence of nets and filters

...nets and filters are the same thing! Well, not exactly. While nets and filters are different mathematical objects *per se*, the notions of convergence that they present are in fact equivalent. This means, broadly, that nets and filters are interchangeable and that replacing one with the other will preserve the nice convergence properties that we proved for arbitrary topological spaces.

To formally show this, we will devise a way to construct a filter from a net and a net from a filter. Then we will show that convergence of one of these implies convergence of the other.

Definition 4.1. *Let $w : D \to X$ be a net in X for a directed set D. Then $\mathcal{F}_w = \{F \subseteq X \mid F \text{ contains a tail of } w\}$ is the* ***derived filter*** *of w.*

Proof. (*that $\mathcal{F}_w$ is a filter*) First, $\emptyset \notin \mathcal{F}_w$. Second, if $F \in \mathcal{F}_w$ contains a tail of w and $F \subseteq G$, then G contains a tail of w, so $G \in \mathcal{F}_w$. Lastly, let $F, G \in \mathcal{F}_w$. We want to show that $F \cap G \in \mathcal{F}_w$. Say

$$T_F = \{w(e) \mid e \geq d_F \in D\}$$

is a tail of w in F and

$$T_G = \{w(e) \mid e \geq d_G \in D\}$$

is a tail of w in G. Take $d = \max\{d_F, d_G\}$. Then the tail $T = \{w(e) \mid e \geq d\}$ is in $F \cap G$. Hence $F \cap G \in \mathcal{F}$. □

So there's a way to produce a filter from a net. Going the other direction will take more care, since we have to specify a directed set for the domain of our net.

Definition 4.2. *Let $\mathcal{F}$ be a filter on X. Equip $\mathcal{F}$ with the relation $F \leq G$ iff $F \supseteq G$. Then $\mathcal{F}$ is a directed set.*

A net $w : \mathcal{F} \to X$ defined by $w(F) \in F$ for all $F \in \mathcal{F}$ is a ***derived net*** *of $\mathcal{F}$.*

Now we also have a way to produce a net from a filter. Here is why we bother to do this two-way conversion.

Theorem 4.1. *A net $w : D \to X$ converges to $x \in X$ iff its derived filter $\mathcal{F}_w$ does.*

Proof. Suppose $w \to x$. Then by definition of net convergence, every open set around x contains a tail of w. Therefore for every open set U containing x, $U \in \mathcal{F}_w$, which means that $F_w \to x$.

Conversely, suppose $\mathcal{F}_w \to x$. Then for every open set U containing x, $U \in \mathcal{F}_w$. Hence U contains a tail of w and $w \to x$. $\square$

Theorem 4.2. *A filter $\mathcal{F}$ converges to $x \in X$ iff every derived net of $\mathcal{F}$ does.*

Proof. Suppose $\mathcal{F} \to x$. Let $w : \mathcal{F} \to X$ be a derived net of $\mathcal{F}$ and U be an open set containing x. By definition of filter convergence, $U \in \mathcal{F}$. We want to show that there is a tail of w in U. For $V \in \mathcal{F}$ and $V \geq U$, i.e. $V \subseteq U$, $w(V) \in V \subseteq U$ by how we defined a derived net. Thus $w \to x$.

Conversely, suppose $\mathcal{F} \not\to x$. We want to show that there is a derived net of $\mathcal{F}$ that does not converge to x. Then there exists an open set U containing x such that $U \notin \mathcal{F}$. Equivalently, for all $F \in \mathcal{F}$, $F \neq U$. Define $w : \mathcal{F} \to X$ such that $w(F) = F \setminus U \in F$. Then w is a derived net of $\mathcal{F}$, but since no tail of w, or rather no point of w, is in U, w does not converge to x. $\square$

Together, these two results affirm that everything convergent nets can characterize, convergent filters can do the same. In particular, these properties of topological spaces—Hausdorff-ness, closure, continuity, and compactness—can be gleaned from analyzing *either* nets *or* filters. Finally, we have that convergent nets *or* convergent filters can completely determine a topological space.

Why do we need two different concepts that can do the same thing? It turns out that while we *can* use nets and filters interchangeably, there are statements that are easier or more elegant to prove with filters than nets, and vice versa. For instance, though we will not touch on Tychonoff's theorem in this article, its proof via filters will be short and sweet.

Addendum

We will explore here why defining subnets as a *cofinal restriction* of the net's domain is not quite adequate. Remember that our goal was to prove this net property: *x is a limit point of a net $w : D \to X$ iff there is a subnet of w converging to x.*

If we did indeed take the definition of subnets to be the one above, then this statement would not be true. Let's look at a counterexample.

Take our space to be $\mathbb{N}^2$ equipped with the topology:

A subset U is open if it does not contain $(0,0)$, *or* if it contains $(0,0)$ and all but finitely many points of all but finitely many columns (imagine we lay $\mathbb{N}^2$ on a grid). In other words, if U contains $(0,0)$, then only a finite number of its columns contain infinite gaps.

Recall that a sequence is a net. In this definition, every subnet of a sequence would be a subsequence. So let's take the sequence winding through every diagonal of our grid, starting from the bottom:

$$(1,1),(2,1),(1,2),(3,1),(2,2),(1,3),(4,1),(3,2),...$$

$(0,0)$ is a limit point of this sequence, since every open set around $(0,0)$ contains infinitely many points of $\mathbb{N}^2$ and thus this sequence.

However, there is no subsequence converging to $(0,0)$. This is because for any subsequence we take, we can produce an open set around $(0,0)$ that throws out infinitely many points of the subsequence, by throwing out either finitely many columns or finitely many points of infinitely many columns. Hence the open set cannot contain a tail of the subsequence.

Since this sequence spans all of $\mathbb{N}^2$, constructing subsequences from other sequences would only herald more restrictions. So this example invalidates our statement and suggests the need to allow subnets to be "enlarged" versions of nets. Here, we can think of each open set around $(0,0)$ to be "super big," and the set of *all* open sets aroundd $(0,0)$ to be "super duper big." While constructing a subsequence converging to $(0,0)$, then, we don't want to be constrained by the ability to include only countably many terms. There are way too many open sets for a normal subsequence to lie in. We can construct a convergent subnet, however, by cherry-picking a term in each open set around $(0,0)$.

For interested readers, this space is called the Arens-Fort space.

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