Surgery Theory

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1 Introduction

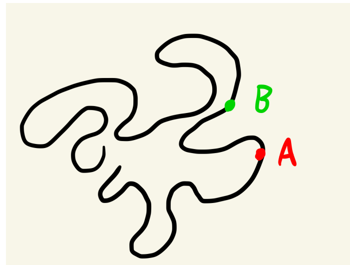
This paper is about surgery theory. As the name suggests, surgery theory deals with the idea of cutting something, that something being a manifold. Surgery itself is a mechanical operation, and the role of surgery theory is to study the effects of the surgery operation on topology and other mathematical structures (including equivalence classes).

Despite the rather mechanical and unintuitive nature of the surgery operation, it has a wide variety of applications. Physicists use it to study topology change under events that “cut” spacetime. Knot theorists use it to classify knots by breaking down their the Seifert surfaces associated with them. Additionally, surgery theory found application in the proof of the Poincare conjecture by Grigori Perelman.

Now, having hyped up surgery theory’s applications, let’s tackle it. This paper takes a geometric approach to explaining surgery theory, relying on diagrams and pictures to build intuition, rather than digging into the notation. We start off by giving a definition and intuition for diffeomorphisms, the language of smooth manifolds, in section 2. We then move onto a discussion and formalization of smooth manifolds themselves in section 3. Having built up the necessary background, we tackle handle decomposition in section 4. This finally leads us to actually formalize surgery and its connection to handle attachment in section 5. The paper finishes with a note on how surgery theory can be used to understand and interpret the Ricci Flow with Surgery technique used by Perelman in his proof of the Poincare conjecture.

2 Diffeomorphisms

Let me pose the following problem to you. You are given the following curve in a non-Euclidean space. Each point has a heat assigned to it. You are given the heat at point A (and only point A) and asked to determine the heat at point B.



Well, this is preposterous! One may have been posed this problem on the unit interval in $\mathcal{R}^1$ and been given the derivative or gradient of the heat at each point. Our problem would then be reduced to $h(B) = h(A) + \int_A^B \frac{dh}{dx} dx$, where $h(x)$ is the heat at point x. However, we don't have a concept of distance here, so how are we supposed to calculate derivatives? Or take integrals?

Ultimately, the concepts of calculus were forged in Euclidean space. They rely on having both a distance metric between and coordinates for points in the space. When we remove these assumptions, we can't use those concepts.

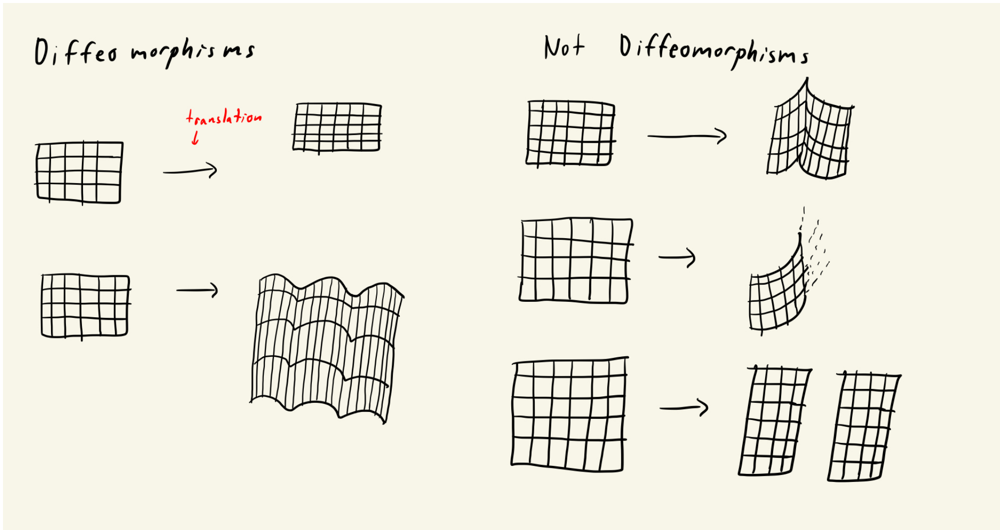
Suppose instead that it is possible for me to parameterize my curve using the unit interval. Say our parameterization function is $f : [0,1] \to S$ where S is our curve. Let's also suppose that f is a homeomorphism, so that the topology of $[0,1]$ is carried to S as well. Thus, we can write our heat function as $h(f(y))$, where $y \in [0,1]$. Now, we can recast our integral as $h(B) = h(A) + \int_A^B \frac{dh}{dx} dx = h(A) + \int_{f^{-1}(A)}^{f^{-1}(B)} \frac{dhf}{dy} dy$. However, we run into an issue because, by chain rule $\frac{dhf}{dy} = \frac{dh}{df}\frac{df}{dy}$. Thus, f must be C^1 for this to work.

We run into this problem over and over again as we try and take higher derivatives. As such, if we want to be able to use the full power of calculus on this curve, we need f to be in C^∞. In other words, f needs to be smooth. With this, we come to the definition of a diffeomorphism.

Definition 1. *A function $f : M \to N$ is a diffeomorphism if it is a bijection, is smooth, and has a smooth inverse f^{-1}*

One will note that because f is smooth, it is differentiable, and because f is differentiable, it is continuous. This also applies for its inverse, so a diffeomorphism is also a homeomorphism. We can think of a diffeomorphism as not only carrying over the topological properties of M to N, but also its differentiability properties.

However, this way of describing diffeomorphisms is very intuitive. Imagine that $M, N \subset \mathcal{R}^3$. A smooth map in this context corresponds to a very "physical" transformation of M into N, one that does not fold or crease M, does not rip it or pinch it infinitely. Some examples of diffeomorphisms are shown below, as well as some non-examples.



Now, recall that two spaces M and N being homeomorphic is an equivalence relation. Similarly, we can show that M and N being diffeomorphic (there exists a diffeomorphism $f : M \to N$) is an equivalence relation.

Lemma 1. *Two spaces M, N being diffeomorphic ($M \sim_{\text{diff}} N$) is an equivalence relation.*

Proof. If $M \sim_{\text{diff}} N$ then there exists a diffeomorphism $f : M \to N$. However, $f^{-1} : N \to M$ is also a diffeomorphism, so $N \sim_{\text{diff}} M$. Thus, $\sim_{\text{diff}}$ is symmetric.

One will also note that the identity map is bijective, smooth and has a smooth inverse, so it is a diffeomorphism between any space M and itself. Thus, $M \sim_{\text{diff}} N$ which implies that $\sim_{\text{diff}}$ is reflexive.

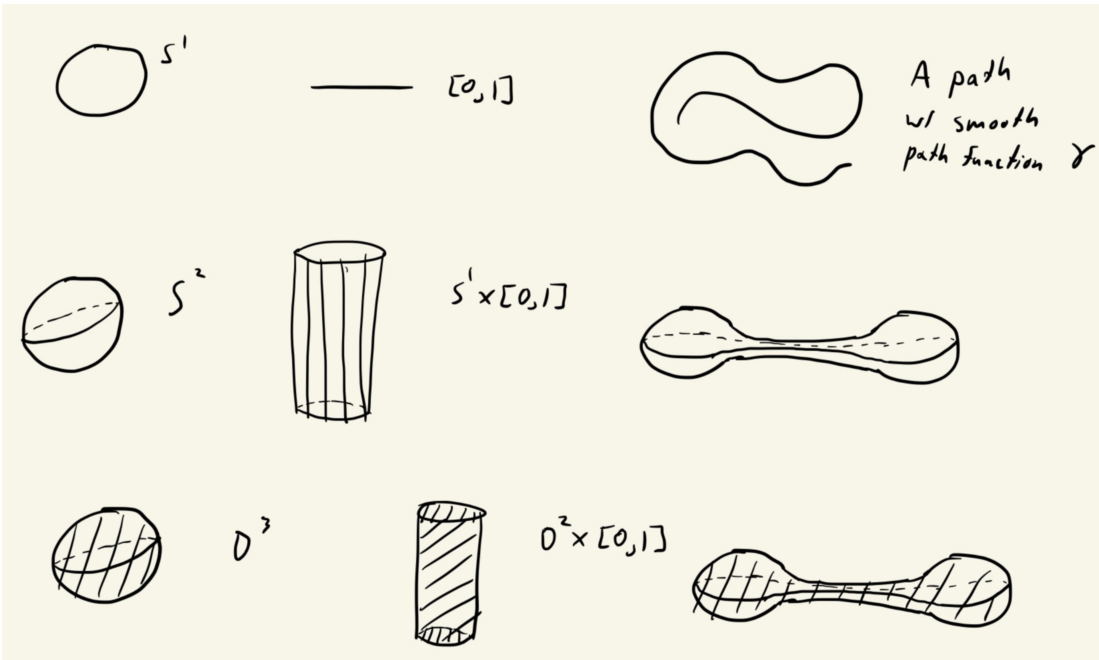
Finally, let's say that $A \sim_{\text{diff}} B$ and $B \sim_{\text{diff}} C$. Thus, we have diffeomorphisms $f : A \to B$ and $g : B \to C$. We know that $gf : A \to C$ is a homeomorphism, we just need to show that it and its inverse are smooth. gf is the composition of smooth functions, and thus is smooth. $(gf)^{-1} = f^{-1}g^{-1}$ is also the composition of smooth functions (f and g have smooth inverses), so it is also smooth. Thus, we conclude that gf is a diffeomorphism. This implies that $A \sim_{\text{diff}} C$, which demonstates that $\sim_{\text{diff}}$ is transitive.

Thus, $\sim_{\text{diff}}$ is an equivalence relation $\square$

Since diffeomorphism induces an equivalence relation, spaces that are diffeomorphic to each other are the same in both a topological and differentiability sense.

3 Smooth Manifolds

Now that we've introduced the idea of a diffeomorphism, we will use it to build up to an understanding of smooth manifolds. Manifolds are an area of math that we have some inherent intuition about, so let's play off of that. Below are some examples of smooth manifolds.



One will note that at any point on these smooth manifolds, I could take a segment of the tangent line or a potion of the tangent plane at a given point and wrap it around the manifold so that all its points intersect with the manifold.

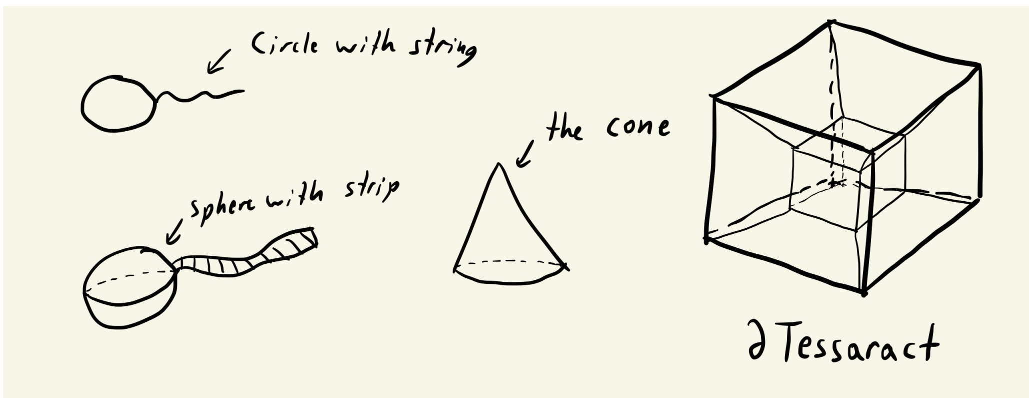
One will also note that the tangent line and tangent plane ideas work for the first and second rows separately, but do not work for the third row. This is because each row contains examples of manifolds that are 1-, 2-, and 3-dimensional. To make this more clear, let's define a smooth manifold formally

Definition 2. *A topological manifold M is a locally Euclidean, Hausdorff space. Locally Euclidean means that if I "zoom in" to a point in M, the space "resembles" Euclidean space topologically. This is formalized by saying that every neighborhood $N \subset M$ is homeomorphic to a subset of a Euclidean space.*

Definition 3. *An n-manifold is a topological manifold such for each point there exists a neighborhood N homeomorphic to a subset of R^n. Saying that each neighborhood N is homeomorphic to a subset of $\mathcal{R}^n$, as opposed to a subset of an arbitrary Euclidean space, further specifies our manifold.*

Definition 4. *A smooth n-manifold is a topological manifold such for each point there exists a neighborhood N diffeomorphic to a subset of R^n. Our manifold is now locally Euclidean in not only a topological sense, but also a differentiable one.*

Thus far, we've gotten some sense of how the above definition works. To add some more nuance to our understanding, let's look at some non-examples of smooth manifolds.

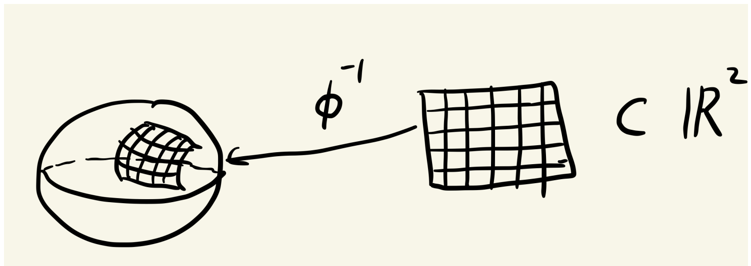


For the "circle with string", take x to be a point at the base of the string. We can see that a neighborhood N of x would contain a 3-pronged shape, which does not exist in R^1. A similar case arises for our "sphere with strip". In another vein, taking a neighborhood of the point at the top of our cone and turning it into a subset of R^2 involves flattening the sharp edge of the cone and so the map cannot be a diffeomorphism. The same idea allows us to show that ∂Tessaract is not a smooth manifold. The tessaract is the 4-dimensional analogue of the filled cube. It's boundary is a hollow cube that increases and decreases in side length along a 4-th axis. Since the tessaract is 4-dimensional, its boundary is a 3-manifold (not necessary smooth). Try finding a smooth map between a neighborhood of one of the corners of one of these hollow cubes within ∂Tessaract and a subspace of R^3. You will find that this neighborhood has a sharp edge that needs to be flattened for any map to be smooth. As such, there is no such smooth map and ∂Tessaract is not a smooth manifold. However, one will note that ∂Tessaract is homeomorphic to S^3 (resembles a sphere that changes radius as you move in the 4th dimension), which is a smooth 3-manifold.

While we are able to discern between smooth and non-smooth manifolds with our current definition, our definition escapes intuition because it's defined point-wise. In order to move towards a more intuitive definition, we introduce charts and atlases.

Definition 5. *Let M be a smooth n-manifold. A chart is a tuple (U, ϕ) where $U \subset M$ is a set and $\phi : U \to \mathcal{R}^n$ is a diffeomorphism between U and $\phi(U) \subset \mathcal{R}^n$.*

We visualize the idea of a chart below for $M = S^2$. We can see that the chart corresponds to a flat sheet that is wrapped around S^2.



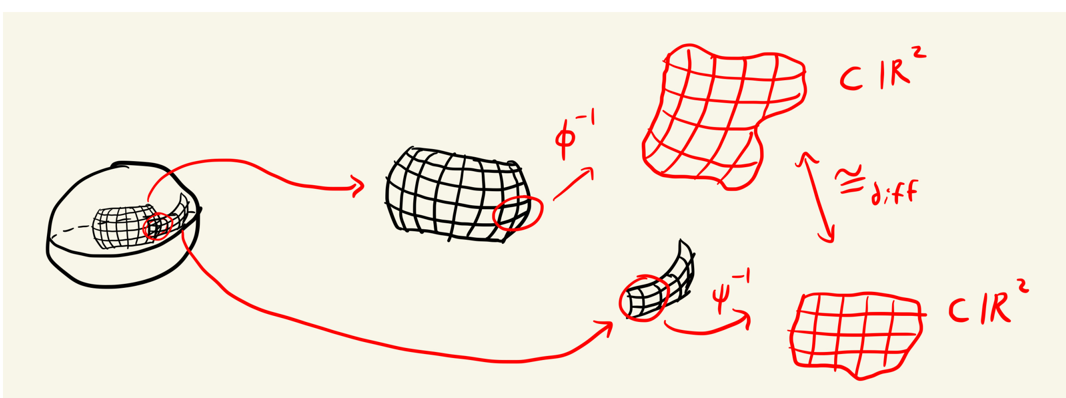
Now, with each of our charts we are able to define the topological and differentiability properties of $U \subset X$. However, we'd like to be able to fully define those properties across our manifold. We can do this by building up a collection of charts called an atlas, which together "cover" our manifold.

Definition 6. *Let X be a smooth n-manifold. An atlas is a collection of compatible charts $\mathcal{A} = \{(U_\alpha, \phi_\alpha) | \alpha \in A\}$ where $\{U_\alpha\}$ is an open cover of X.*

One will note that we've introduced a new term here: compatible. This additional constraint reflects the fact that we may have situations where we have two charts (U, ϕ) and (V, ψ) with U and V overlapping. Given that a chart helps us to define the topological and differentiability properties of subsets of M, it would be problematic if our two charts "disagreed" on these properties. As such, we introduce the notion of compatibility.

Definition 7. *Two charts (U, ϕ) and (V, ψ) are compatible if their transition map $\phi\psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is a diffeomorphism.*

The statement above corresponds to saying that ψ and ϕ agree on both the topological and differentiability properties of $U \cap V$. This is trivially true if $U \cap V = \emptyset$, but if there is some overlap it ensures that the two maps are in agreement. Below we illustrate what this means visually below for $M = S^2$.



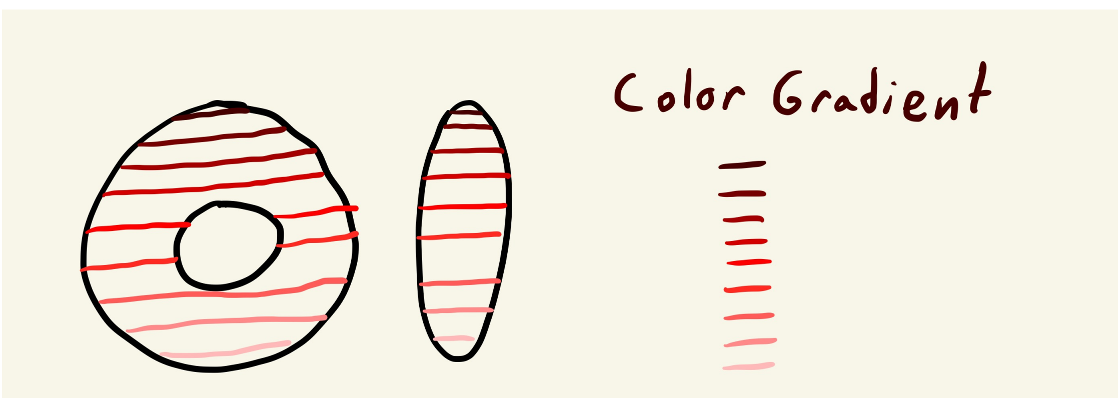
Now that we've built up the definitions of charts and atlases, we can give a far more geometrically intuitive definition of a smooth manifold

Definition 8. *A topological n-manifold is a Hausdorff space with atlas $\mathcal{A}$.*

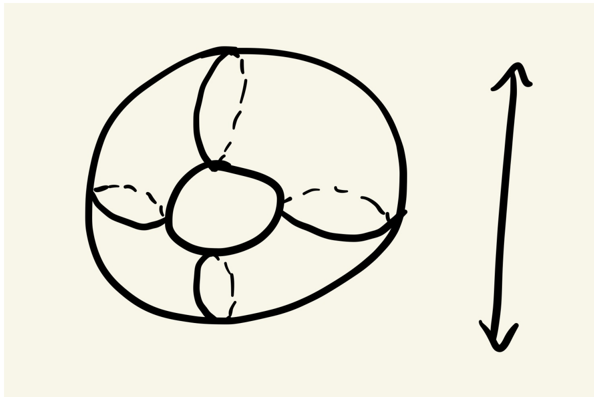
4 Handle Decomposition

In our discussion of smooth manifolds, we showed that they carried all of the topological and differentiability properties of Euclidean space over to a topological manifold M. As such, it makes sense that one should be able to define a smooth, real-valued function $f : M \to \mathcal{R}$, where M is a smooth manifold. A function being "smooth" in this context means that if we "zoom into" the function around a given point, it should resemble a smooth function defined on $\mathcal{R}^n$. This notion can be formalized in terms of charts. Specifically, for any given chart $(U, \phi) \in \mathcal{A}$, the domain-restricted function $f \circ \phi^{-1} : \phi(U) \to \mathcal{R}$ is smooth. These functions are referred to as Morse functions.

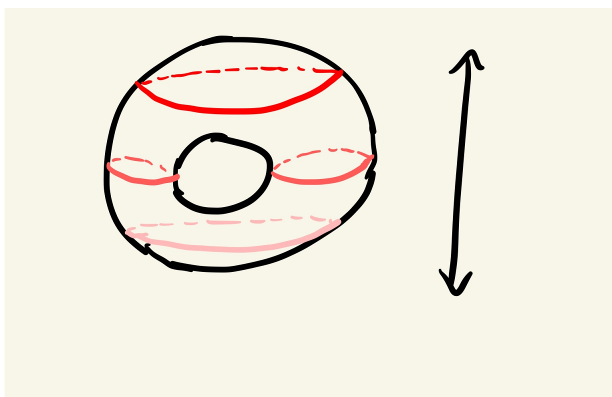
Now, let's try to visualize how a Morse function might look on our manifold M, specifically $M = T^2$. A clearly Morse function (we can see that it is a smooth gradient proceeding up the torus) is visualized in the figure below via contours, where the color of the contours indicates the value of the function (darker red = higher value). Both front and side views of the torus are provided to give the full picture of how the function's value varies over the surface.



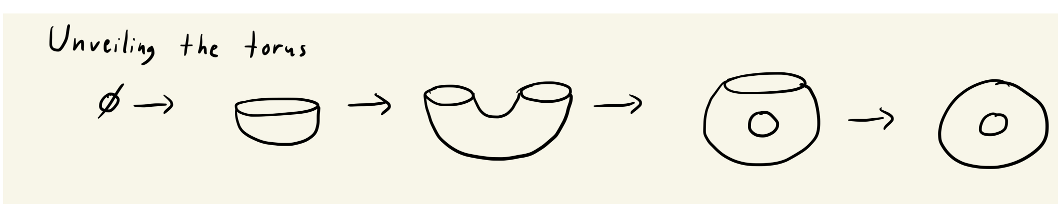
However, we can also think of f as assigning a "height" to a given point $x \in M$. Applying this interpretation to our torus, we get the following visual, where our height axis is shown on the right. Keep in mind that our function being "smooth" means that it pulls our torus out in a very natural way along the height axis. The torus is not broken, pulled or pinched in an discontinuous way at any point.



One interesting property of this "height" interpretation of our manifold is that it allows us to take "slices" of our manifold. Each of these slices correspond to a given height. The slice corresponding to $a \in \mathcal{R}$ is $f^{-1}(a)$. The figure below visualizes a few of these slices on the torus. They are color-coded to emphasize their difference in height.

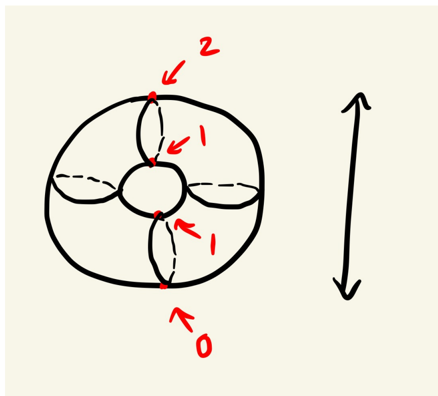


In addition to taking slices, we can also use the height interpretation to "unveil" our manifold, starting from nothing and slowly accumulating slices as we go up in height. The stage of our unveiling associated with a height $a \in \mathcal{R}$ is $M^a = f^{-1}(-\infty, a]$.



We can find something interesting about this unveiling process by comparing M^a and M^b, where $a, b \in \mathcal{R}$ and $a < b$. When do they have the same topology? When is it different?

Upon examination, one finds that there are certain "critical points" $p \in M$ such that if $a < f(p) < b$ then M^a is not homeomorphic to M^b. Critical points are analogous to peaks, troughs, and saddle points for our function f. More specifically, they are points where $\nabla f = 0$. The critical points of our torus are visualized below. Additionally, their "index" is indicated in red. In some sense, the index measures the number of directions in which f is decreasing at a critical point. Convince yourself that the given indices make sense, paying attention to the way the torus curves at each critical point.



Now, we can see that the topology changes at these critical points, but *how* does it change? In order to explore this, we'll need to define the notion of a handle attachment.

Definition 9. *The process of attaching a j-handle to a smooth n-manifold M takes M to $M \cup_f H^j$, where $0 \leq j \leq n$ and $f : S^{j-1} \times D^{n-k} \to \partial M$ is a diffeomorphism. $H^j = D^k \times D^{n-k}$ is a j-handle.*

The idea of a handle attachment seems rather arbitrary, but it can be used to account for a vast array of what are called "topology changes" (events that "edit" the topology of a space). To see this, we will look back at our unveiling procedure. We will be referring to the below figure, which charts how the topology of M^a changes as a increases, as well as how that change can be accounted for using handle attachments.

In order to guide our discussion, we introduce the core concept behind handle decomposition.

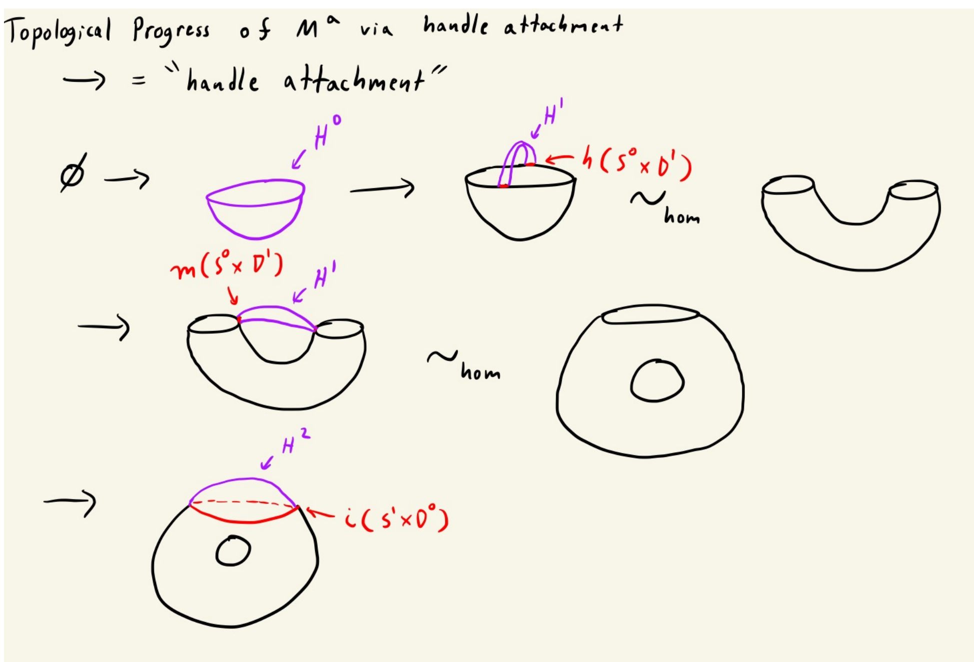
Lemma 2. *Let M be a compact boundaryless manifold. Let $f : M \to \mathcal{R}$ be a Morse function. Assuming that the critical points $\{p_i\} \subset M$ are such that $f(p_1) < f(p_2) < ... < f(p_k)$, and provided $t_0 < f(p_1) < t_1 < f(p_2) < t_2 < ... < f(p_k) < t_k$, $f^{-1}[t_{j-1}, t_j]$ is diffeomorphic to $(f^{-1}(t_{j-1}) \times [0,1]) \cup_g H^{i(p_j)}$ where $i(x)$ gives the index of x. Here g is a diffeomorphism which determines where the handle H^j is attached on ∂M*

We will not give a proof of the above lemma, but instead show how it plays out within our torus unveiling.

We start off with the empty set. We pass our lowest critical point, which has an index of 0. As such, we attach $H^0 = D^0 \times D^2$ to the empty set along $g(S^{-1} \times D^2) = \emptyset$. This means that our partially unveiled torus is now just $H^2 = g(D^0 \times D^2) = g(D^2)$, an embedded disk. Next we pass our second critical point, which has index 1. As such, we attach $H^1 = D^1 \times D^1$ to the empty set along $h(S^0 \times D^1) \in \partial M^a$. This corresponds to attaching a strip from one section of the boundary circle of our embedded disk to the other. We can see that the resulting figure is homeomorphic to the bottom half of the torus.

We now pass the third critical point, which also has an index of 1. As such, we attach $H^1 = D^1 \times D^1$ to our previous figure along $m(S^0 \times D^1)$. The specific handle attachment we do connects the two disconnected circles on the boundary. We can see that the resulting figure is homeomorphic to the torus with just the top missing.

Finally, we pass our final critical point, which has an index of 2. As such, we attach $H^2 = D^2 \times D^0$ to our previous figure along $n(S^1 \times D^0)$. This handle attachment is identical to capping the boundary circle from our previous stage.



One important caveat in this entire process is that if we have two critical points a and b, then $i(a) < i(b) \iff f(a) < f(b)$ (where $i(x)$ gives the index of a point x). In other words, the indexes of critical points are height-ordered. Lower index critical points sit below higher ones. The full explanation of why this must be the case is outside the scope of this paper, but is a key lemma in Morse Theory.

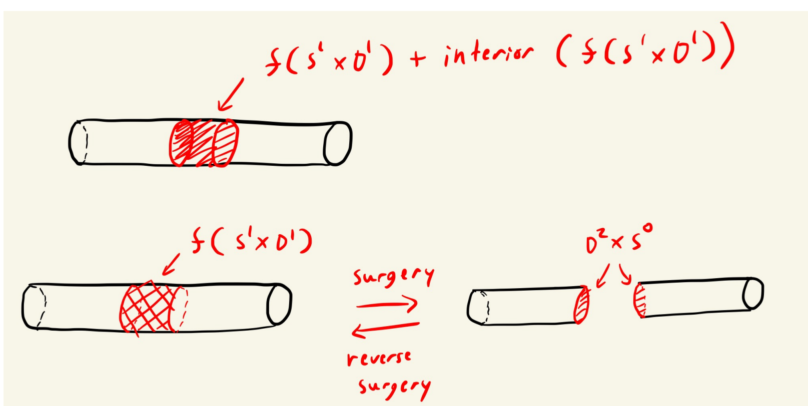
5 Surgery

At this point, we introduce the topic of this paper, surgery theory. The formal definition of the surgery operation is as follows

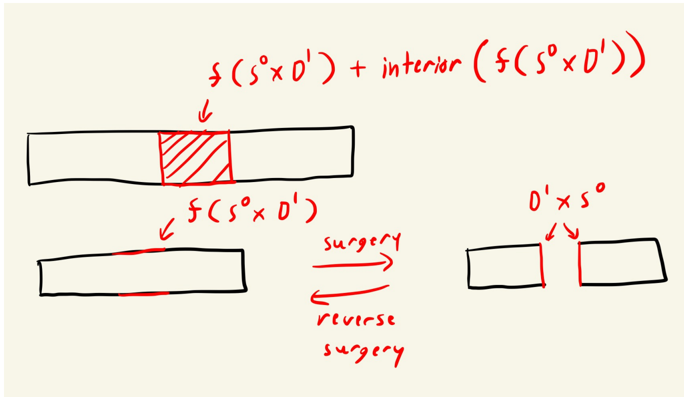
Definition 10. *Given a smooth n-manifold M, $0 \leq k \leq n$, and a diffeomorphism $f : S^k \times D^{n-k} \to M$. The manifold produced by surgery with these parameters is*

$$M' = M - f(S^k \times D^{n-k} - S^k \times S^{n-k-1}) \cup_{f|S^k \times S^{n-k-1}} D^{k+1} \times S^{n-k-1} \quad (1)$$

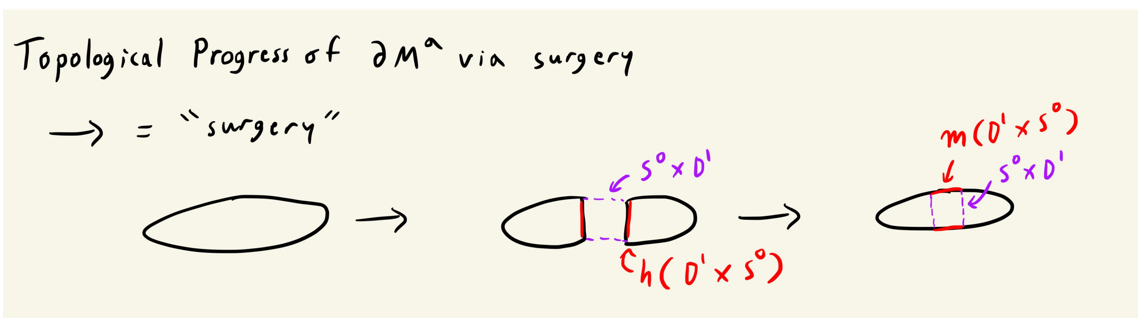
In order to build up our understanding of what this operation looks like, we'll start with a canonical example. Imagine we have a long hollow tube as our manifold M (where we don't care what happens on the ends). We can "embed" a shorter cylinder within this long tube such that the outer edges of the cylinder line up with a subsection of M. This cylinder is diffeomorphic to $D^2 \times D^1$. The basic operation of surgery is replacing the $S^1 \times D^1$ part of the cylinder with its $D^2 \times S^0$ part. Note, however, that $S^1 \times S^0$ is a subset of both $S^1 \times D^1$ and $D^2 \times S^0$, so replacing one with the other doesn't open any holes in our manifold. This is visualized in the figure below. The reverse of this surgery can also be studied, where we replace the $D^2 \times S^0$ part of our cylinder with its $S^1 \times D^1$ part.



This same operation applied to a 2-d strip is shown below



At this point this definition seems rather arbitrary, and one might ask whether surgery comes up in practice. However, we've already seen an example. Recall the way the boundary of our unveiling torus changes as the height a increases. After coming into existence diffeomorphic to S^1 it splits into the disjoint union $S^1 \sqcup S^1$. This disjoint union is rejoined into a form diffeomorphic to S^1 before the boundary disappears. Upon examination, going from $S^1 \to S^1 \sqcup S^1$ and $S^1 \sqcup S^1 \to S^1$ can be accomplished through surgery operations, and yet they are both accounted for by handle attachments on the surface M. The specific process is shown below



In a more general setting, it is possible to prove that a k-handle attachment on the n-manifold M is equivalent to a k-surgery on the boundary ∂M.

Lemma 3. *Let M be an n-manifold. Let $0 \leq k \leq n$ and let $f : S^{k-1} \times D^{n-k} \to \partial M$ be a diffeomorphism.Let $handle(M, k)$ be the manifold produced from M via k-handle attachment with the above parameters. Let $surgery(\partial M, k-1)$ be the manifold produced from ∂M via (k-1)-surgery using the same diffeomorphism. $\partial handle(M, k) = surgery(\partial M, k-1)$.*

Proof. We can see that $handle(M, k) = M \cup_f H^k$. Recall that $H^k = D^k \times D^{n-k}$. We have $\partial H^k = S^{k-1} \times D^{n-k} + D^k \times S^{n-k-1} = B_1 + B_2 = B_1' + B_2' + C$. Here $C =$

$S^{k-1} \times S^{n-k-1}$ is the intersection of $B_1 = S^{k-1} \times D^{n-k}$ and $B_2 = D^k \times S^{n-k-1}$. By removing C from each of our boundary components, we get $B_1' = B_1 - C = S^{k-1} \times D^{n-k} - S^{k-1} \times S^{n-k-1}$ and $B_2' = B_2 - C = D^k \times S^{n-k-1} - S^{k-1} \times S^{n-k-1}$, which are disjoint. When we perform a k-handle attachment on a manifold M, we "cover up" the region $f(S^{k-1} \times D^{n-k}) = f(B_1)$. As such, it is removed from the boundary. However, the boundary of $f(B_1)$ (which is $f(C)$) remains on ∂M because it is also part of $f(B_2)$. Additionally, B_2' is added to the boundary. As such, $\partial handle(M) = (\partial M - f(B_1')) \cup_f (C) B_2$. As this replaces $f(S^{k-1} \times D^{n-k}) = f(B_1)$ with $D^k \times S^{n-k-1} = B_2$ along $f(S^{k-1} \times S^{n-k-1}) = f(C)$, we have $\partial handle(M, k) = \partial M - f(S^{k-1} \times D^{(n-1)-(k-1)} - S^{k-1} \times S^{(n-1)-(k-1)-1}) \cup_f (D^{(k-1)+1} \times S^{(n-1)-(k-1)-1}) = surgery(\partial M, k-1)$. $\square$

It's also important to note that since ∂M is (n-1)-dimensional, every surgery operation on the boundary can be accounted for by a handle attachment on the manifold.

6 Ricci Flow with Surgery

Now that we've built up some background in Surgery and Morse theory, let's attempt to interpret Perelman's proof of the Poincare conjecture using Ricci flow with surgery. Let's start off by stating the Poincare conjecture

Conjecture 1. *All simply-connected, closed 3-manifolds are homeomorphic to* S^3

There are a few things to unpack here. First, for a manifold to be simply-connected, it must both be path connected and have all loops contract to a point. In practice, this means there are no "holes" in our manifold (think the torus). Further, a closed manifold has no boundary and is compact. For some context on what S^3 is, think back to our discussion about ∂Tessaract

The conjecture itself seems rather tame, but the methods it took to prove it are not. I will not be able to do them justice here, but I will try to give at least an overview. To start off, the following is the formal definition of Ricci flow

Definition 11. *Let M be a smooth manifold and let $(a, b) \subset \mathcal{R}$ be an open interval (denotes time in this context). Under Ricci flow, at time $t \in (a, b)$, we have*

$$\frac{\partial}{\partial t} g_t = -2\mathrm{Ric}^{g_t} \tag{2}$$

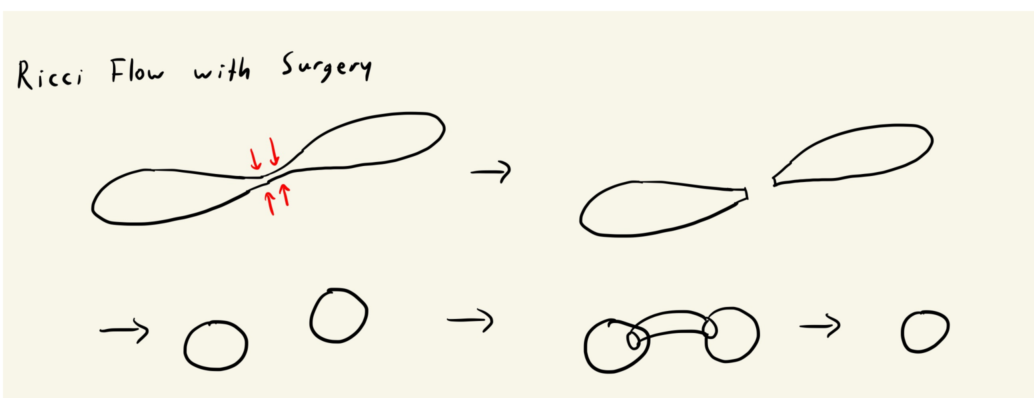
where g_t is the Riemannian metric and Ric is the Ricci curvature.

Full descriptions of Riemannian metrics and Ricci curvature are outside the scope of this paper. However, the Riemannian metric can be seen to denote the distance between points along the manifold, while the Ricci flow captures information about how the manifold curves. One can think of Ricci curvature as analogous to heat flow, except the output is not a change in value, but a change in shape. However, what will be most useful for our purposes is to describe what Ricci flow does to a shape (in ideal situations).

Note the negative sign in front of the Ricci curvature. This means that, in some sense, if the Ricci curvature is large, then the distance between points will shrink, while if it is negative, the distance between points will grow. All along, the curvature of the points is changing

It has been found that Ricci flow collapses certain 3-manifolds to a point. Perelman found that the shape these manifolds have right before it became a point was S^3. As such, we can run back the process and show that those manifolds are homeomorphic to S^3. However, Perelman ran into some issues generalizing this approach, as not all 3-manifold collapse to a point. One of those issues is the occurrence of "pinching points". One of the varieties of pinching points is shown below. At these points, the Ricci flow collapses down to an infinitely thin bottleneck, leaving the rest of the manifold at the same size, which makes it pretty hard to argue that these manifolds are homeomorphic to S^3.

In order to combat this problem, Perelman had the idea to "snip" these points off using surgery, then let the process continue on both sides of the pinching point separately. If another pinching point occurred later on, it would also be snipped. After some time, all the "blobs" this produces become points. We then pull them back to being S^3. Proceeding pairwise, we do reverse surgeries of these S^3. As one might recall from an introductory topology course, each of these reverse surgeries is equivalent to taking a connected sum of the two manifolds being put together, and $S^3 \# S^3 = S^3$, so the procedure recursively builds back up to S^3. The whole process is visualized in the figure below.



In the current description of Ricci flow with surgery, the Ricci flow and surgery seem like two separate actions, trading off with one another. However, with our previous conversation, it is possible to recast the entire process in terms of handle attachments. Think of a 4-dimensional manifold composed of slices, where each slice is the Ricci flow at a given time. Since both the original manifold and S^3 are simply connected, we can retract them to a point at both ends of our 4-manifold so that it is compact and boundaryless. As such, we can break our 4-manifold up into a handle decomposition using a Morse function. In this analogy, our pinching points become critical points (analogous to saddle points). Just as with our torus, handle attachments accomplish surgeries (splitting two 3-manifolds) and reverse surgeries (connected summing 3-spheres).

We have gone from what seemed like a rather arbitrary process to a complex and, frankly, beautiful 4-dimensional object that fully describes a physical process. This realization points us to one of the core strengths of surgery theory: its ability to elegantly describe physical processes which consist of topology change events. The portion of our 4-manifold that goes from the original 3-manifold's slice to the S^3 slice is called a cobordism between the original 3-manifold and S^3, a smooth manifold with a disjoint union of the original 3-manifold and S^3 as its boundary. For interested readers, cobordism theory is the study of these cobordisms, which are used to describe topology change events in many areas of math and science, including knot theory and quantum gravity.

7 References

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