1 Motivation

After a long hard semester in Analysis I, one marked by numerous dates with Walter Rudin in Butler Library, a student of mathematics should feel not only defeated but also accomplished. The passing of the course marks an acquaintance with proofs, a mastery of epsilons and deltas, the start of the transition to what many mathematicians would call "real mathematics." But then in the student's topology class, what should be the light at the end of the analytical tunnel, the metric is promptly discarded to allow for a more abstract definition of a topology without much justification. A natural question arises: Why get rid of the metric which proved so useful the previous semester? It turns out that mathematics professors aren't sadists. Rather, certain topological spaces simply cannot be metrized: there exists no metric that will allow us to properly express that topology. Even better, thanks to two mathematicians named Jun-iti Nagata and Yuri Mikhailovich Smirnov (who proved the same result independently within a year of one another) we know the exact conditions which are both necessary and sufficient for a topological space to be metrizable. This result is, perhaps unsurprisingly, known as the **Nagata-Smirnov Theorem**.

2 Prerequisites

Before we prove the Nagata-Smirnov Theorem (and in the process the perhaps even more famous Urysohn metrization theorem) I must introduce a few formal definitions. The first two definitions describe the countability axioms.

Definition 2.1 (First Countability Axiom)**.** A space X is said to have a **countable basis at x** if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space X satisfies the **first countability axiom** and is said to be **first countable** when each of its points has a countable basis.

Definition 2.2 (Second Countability Axiom)**.** A space X is said to satisfy the **second countability axiom** and is said to be **second countable** if it has a countable basis for its topology.

Broadly, second countability is a sort of tameness assumption. It's a nice property that guarantees the topological space we are working in is, to some extent, well-behaved. Note that second countability is a stronger condition then first countability. If a space has a countable basis $\mathcal{B}$ then the subset of $\mathcal{B}$ consisting of those basis elements containing the point x is a countable basis at x. In fact, second countability is such a strong condition that not even all metric spaces satisfy it.

Example 1. The real line $\mathbb{R}$ is second countable. The collection of all open intervals of the form (a, b) with rational endpoints forms a countable basis. Similarly, $\mathbb{R}^n$ is second countable. The collection of all products of open intervals with rational endpoints forms a countable basis.

Example 2. The Sorgenfrey line $\mathbb{R}_l$ is **not** second countable. For any point $x \in \mathbb{R}$, $[x, \infty)$ is an open set. Thus, by the definition of a basis, for any $x \in \mathbb{R}$ we can choose a basis element $B \in \mathcal{B}$ of the form $[x, y)$ where $y > x$. These basis elements are distinct so $|\mathcal{B}| \geq \mathbb{R}$ which implies the basis is uncountable.

Definition 2.3. A subset A of a space X is said to be **dense** in X if $\bar{A} = X$.

Example 3. The set of rationals $\mathbb{Q}$ is dense in the reals $\mathbb{R}$.

The intuition here is that a subset A is dense in a space X if we can get arbitrarily close to all points in X via points in A. The next set of definitions concerns three separation axioms. You are probably already familiar with the first one.

Definition 2.4 (Normal)**.** Assume that one-points sets are closed in X. Then X is said to be **normal** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets U and V containing A and B respectively.

Definition 2.5 (Regular)**.** Assume that one-points sets are closed in X. Then X is said to be **regular** if for each pair of a point x and a closed set B disjoint from x, there exist disjoint open sets U and V containing x and B respectively.

Definition 2.6 (Hausdorff)**.** A space X is said to be **Hausdorff** if for each pair of distinct points x, y in X, there exist disjoint open sets U and V containing x and y respectively.

Because we have assumed that one-point sets are closed, normal spaces are regular, and regular spaces are Hausdorff; I have listed the definitions in order of decreasing strength. This assumption is crucial for this relation to hold: a two-point space with the indiscrete topology, for instance, satisfies the other components of the definitions of regularity and normality but is not Hausdorff. It should be clear why these are called separation axioms: we are separating some combination of points and sets. See Figure 1.

Example 4. The Sorgenfrey line $\mathbb{R}_l$ is normal. One-point sets are closed since the topology of $\mathbb{R}_l$ is finer than that of $\mathbb{R}$. Now suppose A and B are disjoint closed sets. Then for each point $a \in A$ choose a basis element $[a, x_a)$ not intersecting B. Similarly, for each point $b \in B$ choose a basis element $[b, x_b)$ not intersecting A. Then

$$U = \bigcup_{a \in A} [a, x_a) \text{ and } V' = \bigcup_{b \in B} [b, x_b)$$

are disjoint open sets containing A and B respectively.

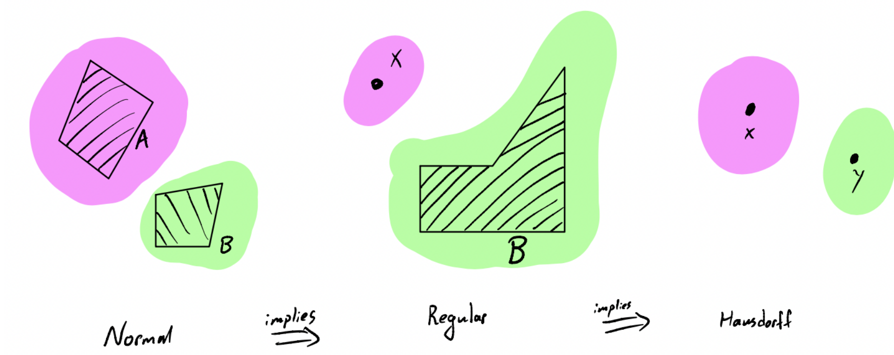

Figure 1: Separation Axioms

I now introduce an alternative characterization of normal spaces that will become useful when proving the Urysohn Lemma. It will allow us to construct a sequence of open sets that start off looking like one type of set and gradually change to look like another.

Lemma 2.1 (Alternative characterization of normal spaces)**.** *X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$.*

Proof. Suppose first that X is normal and that both A and U are given. Let $B = X - U$ which is closed since U is open. Because X is normal, there exist disjoint open sets V and W containing A and B respectively. $\bar{V}$ is disjoint from B because if $y \in B$, the set W is a neighborhood of y disjoint from V implying $y \notin V$. Therefore $\bar{V} \subset X - B = U$.

To prove the converse, assume that disjoint closed sets A and B are given. Let $U = X - B$ which is open. Then $A \subset U$ and by hypothesis there is an open set V containing A such that $\bar{V} \subset U$. Note also that $B = X - U \subset X - \bar{V}$. The open sets V and $X - \bar{V}$ are disjoint closed sets containing A and B respectively. So X is normal. □

Lemma 2.2 (Alternative characterization of regular spaces)**.** *X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V} \subset U$*

Proof. The proof is nearly identical to the one above: simply replace "the set A" by "the point x" throughout. □

Theorem 2.3. Every metrizable space is normal

Proof. Let X be a metrizable space with metric d, and let A and B denote disjoint closed sets. For each $a \in A$ choose $\epsilon_a > 0$ sufficiently small so that $B(a, \epsilon_a)$ does not intersect B. Similarly, for each $b \in B$ choose $\epsilon_b > 0$ sufficiently small so that $B(b, \epsilon_b)$ does not intersect A. See figure 2. Now define

$$U = \bigcup_{a \in A} B(a, \epsilon_a/2) \text{ and } V = \bigcup_{b \in B} B(b, \epsilon_b/2)$$

Then U and V are open sets containing A and B respectively. Moreover, they are disjoint. To see this, suppose, for contradiction, that $z \in U \cap V$. Then

$$z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$$

for some $a \in A$ and some $b \in B$. Then by applying the triangle inequality we obtain that $d(a, b) < d(a, z) + d(z, b) = \epsilon_a/2 + \epsilon_b/2$. If $\epsilon_a \leq \epsilon_b$ then $d(a, b) < \epsilon_b$ so the ball $B(b, \epsilon_b)$ contains the point a. Similarly, if $\epsilon_b \leq \epsilon_a$ then $d(a, b) < \epsilon_a$ so the ball $B(a, \epsilon_a)$ contains the point b. Either case is a contradiction and so U and V must be disjoint. A and B were arbitrary disjoint closed sets and U and V are disjoint open sets containing A and B respectively X must be normal. □

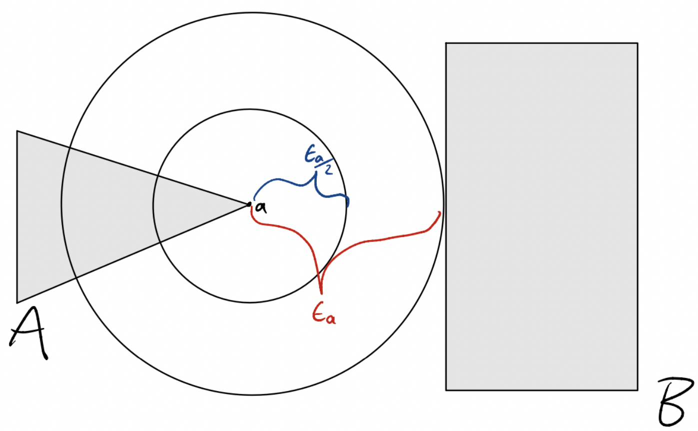

Figure 2: Every Metrizable Space is Normal

Example 5. Although the Sorgenfrey line $\mathbb{R}_l$ is normal, the Sorgenfrey plane $\mathbb{R}_l^2$ (the product of two copies of the Sorgenfrey line) is not. Then the Sorgenfrey plane is not metrizable by the contrapositive of the lemma. For a proof of why the Sorgenfrey plane is not normal see Arnaud and Rudnicki's 2013 paper *Some Properties of the Sorgenfrey Line and the Sorgenfrey Plane.*

3 Urysohn Lemma

Don't be fooled by the word "lemma." This is the first real proof of this paper, and it requires a substantial amount of work. It asserts the existence of certain real-valued continuous functions on normal spaces and will be crucial when proving the Urysohn metrization theorem. With that said, then, let's get on with the proof.

Lemma 3.1 (Urysohn Lemma)**.** Let X be a normal space. Let A and B be two disjoint closed subsets of X, and let $[a,b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \to [a,b]$ such that $f(x) = a$ for all $x \in A$ and $f(x) = b$ for all $x \in B$.

Proof. Assume for now that the interval in question is the interval $[0,1]$. The general case follows then follows easily. The proof is quite long so I'll break it up into steps.

Step 1. Because X is normal we can find an open set U such that $A \subset U$ and $\bar{U} \subset X - B$. The idea here is to construct an infinite number of sets such that the sets start off looking like U and end up looking like $X - B$. More specifically, let P denote the set of all rationals in the interval $[0,1]$. For each $p \in P$ we will define an open set $\bar{U}_p$ such that $\bar{U}_p \subset U_q$ whenever $p < q$. Thus, the sets U_p will be ordered by inclusion the same way their subscripts will suggest.

Because P is countable we can use induction to define the sets U_p. Arrange the elements of P in an infinite sequence such that the numbers 0 and 1 are the first two elements. Let P_n denote the first n rationals in the sequence. Let $U_1 = X - B$ which is open. Since X is a normal space and A is a closed set contained in U_1 (A and B are disjoint), we may choose an open set U_0 such that $A \subset U_0$ and $\bar{U}_0 \subset U_1$. Thus we have defined U_p for all $p \in P_2$.

Now we proceed by induction. Assume that U_p is a defined open set for all $p \in P_n$ and that these sets satisfy the condition $p < q \Longrightarrow \bar{U}_p \subset U_q$. Let r denote the next term in the sequence. We wish to define U_r such that the condition still holds for $P_{n+1} = P_n \cup \{r\}$. Let p denote the largest number in P_n smaller than r, and let q denote the smallest number in P_n larger than r. U_p and U_q are already defined and $\bar{U}_p \subset U_q$ by the inductive hypothesis. Because X is a normal space we can construct an open set U_r such that $\bar{U}_p \subset U_r$ and $\bar{U}_r \subset U_q$.

I now show that the above condition is also satisfied for all pairs of elements in P_{n+1}. First, suppose that both elements of P_{n+1} are also elements of P_n. Then the condition is satisfied by the inductive hypothesis. Now suppose instead that one of the elements is r. Then either $s \leq p$ in which case $\bar{U}_s \subset \bar{U}_p \subset U_r$ or $s \geq q$ in which case $\bar{U}_r \subset U_q \subset U_s$. So the condition is satisfied for P_{n+1} as well.

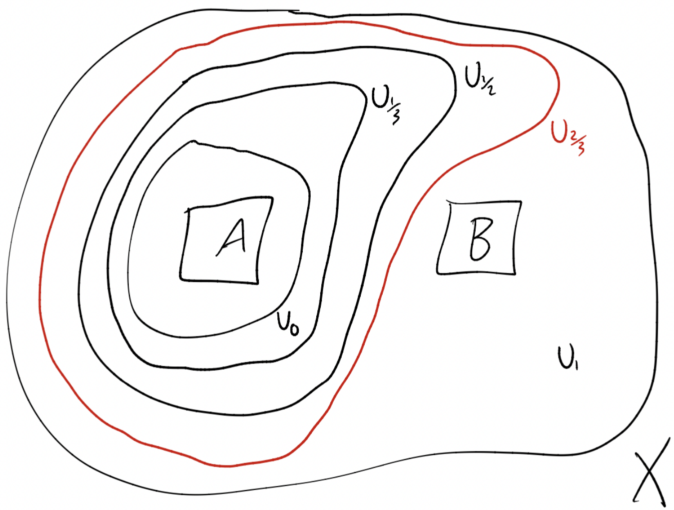

Figure 3: Inserting $U_{2/3}$ when $P = \{0, 1, 1/2, 1/3, 2/3, ...\}$

Step 2. We extend the definition of U_p from all rationals in $[0, 1]$ to all rationals in $\mathbb{R}$. We do this by defining $U_r = \varnothing$ if $r < 0$ and $U_r = X$ if $r > 1$. The condition still holds for all $p \in \mathbb{R}$ since $\varnothing = \bar{U}_p \subset U_q$ whenever $p < 0$ and $\bar{U}_p \subset U_q = X$ whenever $q > 1$.

Step 3. Given a point $x \in X$ define $\mathbb{Q}_x$ to be the set of rational numbers whose corresponding open sets U_p contain x. More formally let $\mathbb{Q}_x = \{p | x \in U_p\}$. This set contains no number less than 0 since $U_r = \varnothing$ for $r < 0$ and this set contains all numbers greater than 1 since $U_r = X$ for $r > 1$. Then the function $f(x) = \inf \mathbb{Q}_x = \inf\{p | x \in U_p\}$ is bounded below by 0 and above by 1

Step 4. Now we show that f is the desired function. If $x \in A$ then $x \in U_p$ for all $p \geq 0$ so $f(x) = \inf \mathbb{Q}_x = 0$. If $x \in B$ then $x \notin U_p$ for all $p \leq 1$ so $f(x) = \inf \mathbb{Q}_x = 1$ since $\mathbb{Q}_x$ consists of all rational numbers greater than 1.

The harder part is proving that f is continuous. To see this we begin by making two observations. First, $x \in \bar{U}_r \implies f(x) \leq r$. If $x \in \bar{U}_r$ then $x \in U_s$ for all $s > r$ so $\mathbb{Q}_x$ contains all rational numbers greater than r and by definition $f(x) = \inf \mathbb{Q}_x \leq r$. Second, note that $x \notin U_r \implies f(x) \geq r$ by similar logic. If $x \notin U_r$ then $x \notin U_s$ for all $s < r$. Therefore, $\mathbb{Q}_x$ contains no rational numbers less than r and by definition $f(x) = \inf \mathbb{Q}_x \geq r$.

Now we prove continuity of f. Given a point x_0 of X and an open interval (c, d) in $\mathbb{R}$ containing $f(x_0)$ we wish to find a neighborhood U of x_0 such that $f(U) \subset (c, d)$. To do so we choose rational numbers p and q such that $c < p < f(x_0) < q < d$ and consider the open set $U = U_q - \bar{U}_p = U_q \cap (X - \bar{U}_p)$.

First we show that $x_0 \in U$. Note that $f(x_0) > p$ implies $x \notin \bar{U}_p$ by the contrapostive of the first observation. Similarly, $f(x_0) < q$ implies $x_0 \in U_q$ by the contrapositive of the second observation. Second, we show that $f(U) \subset (c, d)$. Let $x \in U$. Then $x \in U_q \subset \bar{U}_q$ so $f(x) \leq q$ by our first observation. Similarly, $x \notin U_p$ so $f(x) \geq p$ by our second observation. See figure 4. Thus, $f(x) \in [p, q] \subset (c, d)$ and $f(U) \subset (c, d)$. So f is continuous.

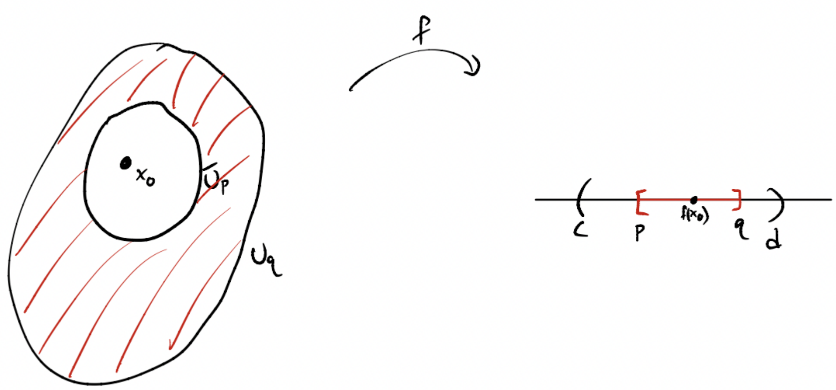

Figure 4: Continuity of f

Generalization We now consider the general case of a closed interval $[a, b]$. First construct a function $f : X \to [0, 1]$ such that $f(x) = a$ for all $x \in A$ and $f(x) = b$ for all $x \in B$ using the four steps above. Now let $h = g \circ f$ where $g(x) = (b - a)x + a$. g is affine and thus continuous. h is continuous as well as the composition of two continuous functions. Now observe $h(x) = g(f(x)) = g(0) = a$ for all $x \in A$ and $h(x) = g(f(x)) = g(1) = b$ for all $x \in B$. $\square$

4 Urysohn Metrization Theorem

We now get to the well-known Urysohn Metrization Theorem. This theorem provides conditions under which a topological space is metrizable, and its proof uses results on metric topologies as well as results on the countability and separation axioms proved earlier. The proof will soon be generalized when I present the Nagata-Smirnov metrization theorem. Before delving into the theorem itself, I review the definitions of the uniform metric and an embedding. For the uniform metric, we essentially define the distance between two points to be the largest distance between any two corresponding coordinates of those two points. The definition of an embedding should be review.

Definition 4.1 (Uniform Metric on $\mathbb{R}^J$)**.** Given an index set J, and given points $\mathbf{x} = (x_j)_{j \in J}$ and $\mathbf{y} = (y_j)_{j \in J}$ in $\mathbb{R}^J$ we define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by the equation

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) | \alpha \in J\}$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$ and defined by the equation

$$\bar{d}(x, z) = \min\{|x - y|, 1\}$$

We call $\bar{\rho}$ the **uniform metric** on $\mathbb{R}^J$, and the topology it induces is called the **uniform topology**.

Definition 4.2 (Embedding)**.** Suppose $f : X \to Y$ is an injective continuous map. If $f' : X \to f(X)$ obtained by restricting the range of f is a homeomoporhism, we say that the map f is an **embedding** of X in Y.

Theorem 4.1 (Urysohn Metrization Theorem)**.** Every regular space X with a countable basis is metrizable.

Proof. We shall show that X is metrizable by embedding X in a in a metrizable space Y; that is, by showing X is homeomorphic with a subspace of Y.

Step 1. We begin by proving that there exists a countable collection of continuous functions $f_n : X \to [0, 1]$ having the property that given any point x_0 of X and any neighborhood U of x_0, there exists an index n such that f_n is positive at x_0 and vanishes outside U.

Let $\mathcal{B} = \{B_n\}$ be a countable basis. For each pair n, m of indices where $\bar{B}_n \subset B_m$, apply the Urysohn lemma to choose a continuous function $g_{n,m} : X \to [0, 1]$ such that $g_{n,m}(\bar{B}_n) = \{1\}$ and $g_{n,m}(X - \bar{B}_m) = \{0\}$. We now show that the collection $\{g_n, m\}$ satisfies the above property. Given x_0 and a neighborhood

U of x_0 we can choose a basis element B_m such that $x \in B_m \subset U$. Using the alternative characterization of regular spaces from section 2, we can choose B_n so that $x_0 \in B_n$ and $\bar{B}_n \subset B_m$. Then n, m is a pair of indices for which the function $g_{n,m}$ is defined. Moreover this function is positive at x_0 and vanishes outside U. Additionally, because the collection $\{g_{n,m}\}$ is indexed with a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$ it is countable. Reindexing this set with the positive integers gives us the defined collection of functions f_n.

Step 2. Given the functions f_n from step 1, we construct a map $F : X \to \mathbb{R}^\omega$ where $\mathbb{R}^\omega$ has the product topology and the map F is define by the following equation

$$F(x) = (f_1(x), f_2(x), ...)$$

We assert that F is an embedding.

F is clearly continuous in the product topology since each f_n is continuous. Additionally, F is injective because given $x \neq y$ there exists an index n such that $f_n(x) > 0$ and $f_n(y) = 0$ which implies $F(x) \neq F(y)$. To see this, recall that X is regular and thus Hausdorff so there exists a neighborhood U of x distinct from y. Then by step 1 there exists a function f_n such that $f_n(x) > 0$ and $f_n(y) = 0$ because $y \notin U$.

Now we must show that F is a homeomorphism of X onto its image $F(X) \subset \mathbb{R}^\omega$. We already know F is a continuous bijection onto its image (surjectivity comes for free by definition) so we now only need to show that the map F is open. To do this, we will show that for any open set U in X and $z_0 \in F(U)$, we can find an open set W in $F(X)$ such that

$$z_0 \in W \subset F(U)$$

Let x_0 denote the point of U such that $F(x_0) = z_0$. Choose an index N such that $f_N(x) > 0$ and $f_N(X - U) = \{0\}$. Then the set

$$V = \pi_N^{-1}((0, +\infty)) \subset \mathbb{R}^\omega$$

is open since $(0, +\infty)$ is open in $\mathbb{R}$ and the projection mappings are continuous. Now let $W = V \cap F(X)$ which is open in $F(X)$ by the definition of the subspace topology. We assert that $z_0 \subset W \subset F(U)$. First, $z_0 \in W$ because

$$\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0$$

Second, $W \subset F(U)$. If $z \in W$ then $z = F(x)$ for some $x \in X$, and $\pi_N(z) \in (0, +\infty)$. Since $\pi_N(z) = \pi(f_N(F(x)) = f_N(x)$ and $f_N(X - U) = \{0\}$, it must be the case that $x \in U$. Then $z = F(x) \subset F(U)$ as desired. See figure 5. So the map F is open, and F is an embedding of X in $\mathbb{R}^\omega$.

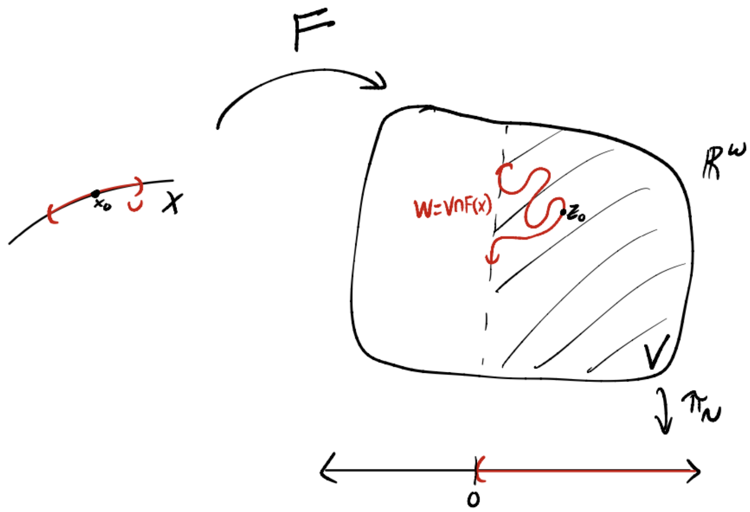

Figure 5: Embedding

□

We can actually generalize step 2 of the theorem with only a little work. This stronger result will become necessary when we prove the Nagata-Smirnov theorem: it can be applied to $\mathbb{R}^J$ even when J is not countable.

Theorem 4.2 (Embedding Theorem)**.** Let X be a space in which one-point sets are closed. Suppose that $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous function $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that for each point x_0 of X and each neighborhood U of x_0 , there is an index α such that f_α is positive and x_0 and vanishes outside U. Then the function $F : X \to \mathbb{R}^J$ defined by

$$F(x) = (f_\alpha)_{\alpha \in J}$$

is an embedding of X in $\mathbb{R}^J$. If f_α maps X into $[0, 1]$ for each α, then F embeds X into $[0, 1]^J$

Proof. The proof is essentially a copy of step 2 of the above proof but we replace "n" by "α" and "$\mathbb{R}^\omega$" by "$\mathbb{R}^J$" throughout. We need one-point sets in X to be closed so that given $x \neq y$ there is always an index α such that $f_\alpha(x) \neq f_\alpha(y)$. $\square$

Example 6. A topological space X is called locally Euclidean if there is a positive integer n such that every point in X has a neighborhood homeomorphic to $\mathbb{R}^n$. A locally Euclidean space that is also Hausdorff is called a topological manifold. Then every topological manifold is automatically regular and every second-countable manifold is metrizable.

5 Preliminaries

We've just proved the Urysohn Metrization Theorem. It's an important result, providing conditions under which a space is metrizable, but mathematicians are greedy. Even better than a sufficient condition would be a condition that is both necessary and sufficient for metrizability. To do so will require some new definitions that we haven't yet formulated. In this section, I'll introduce the definition of a **locally finite** topological space and prove some basic results. Following this, we'll be ready to prove the Nagata-Smirnov Metrization Theorem and fully characterize when a space is metrizable.

Definition 5.1 (Locally Finite)**.** Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be **locally finite** in X if every point of X has a neighborhood that intersects only finitely many elements of $\mathcal{A}$

Definition 5.2 (Countably Locally Finite)**.** A collection $\mathcal{B}$ of subset of X is said to be **countably locally finite** if $\mathcal{B}$ can be written as the countable union of collections $\mathcal{B}_n$, each of which is locally finite.

Definition 5.3 (Refinement)**.** Let $\mathcal{A}$ be a collection of subsets of the space X. A collection $\mathcal{B}$ is said to be a **refinement** of $\mathcal{A}$ if for each element $B \in \mathcal{B}$, there is an element $A \in \mathcal{A}$ containing B. If the elements of $\mathcal{B}$ are open sets, we call $\mathcal{B}$ an **open refinement** of $\mathcal{A}$; similarly, if they are closed sets we call $\mathcal{B}$ a **closed refinement**.

Similar to second countability, local finiteness is a condition under which topological spaces are reasonably well-behaved. The same can be said about space that are countably locally finite though of course this classification is more broad. A refinement allows us to look at a smaller collection of sets "covered" by the original collection. It is often more convenient to work with this refinement than the original collection.

Example 7. The collection of intervals

$$\mathcal{A} = \{(n, n+2) | n \in \mathbb{Z}\}$$

is locally finite in the topological space $\mathbb{R}$ since for any $x \in \mathbb{R}$, the neighborhood $(x-1, x+1)$ intersects at most 4 elements of $\mathcal{A}$. On the other hand the collection

$$\mathcal{B} = \{(0, 1/n) | n \in \mathbb{Z}_+\}$$

is not since any neighborhood of $0 \in \mathbb{R}$ intersects infinitely many elements of $\mathcal{B}$.

Example 8. Any collection of sets $\mathcal{B}$ is a refinement of $\{X\}$ because for all $B \in \mathcal{B}$ we have $B \subset X$.

The following definition and theorem will be quickly stated but I won't dwell much on them since this is a paper focusing on topology not set theory. They will be used in the proof of the following lemma.

Definition 5.4. A set A with an order relation $<$ is said to be **well-ordered** if every nonempty subset of A has a smallest element.

Theorem 5.1 (Well-ordering theorem)**.** If A is a set, there exists an order relation on A that is a well-ordering.

Proof. See Zermelo's 1904 paper, *Beweis, daß jede Menge wohlgeordnet werden kann* □

This theorem was startling but the only controversial step in the proof was a construction involving the axiom of choice. Consequently, several mathematicians rejected the axiom of choice. Most mathematicians today accept the axiom of choice, but here's an example to illustrate why several mathematicians were hesitant to believe it.

Example 9. There exists an order relation on $\mathbb{R}$ that is a well-ordering by the well-ordering theorem. This means that every non-empty subset of $\mathbb{R}$ would have a least element under the well-ordering.

Despite this discomforting example we can use the well-ordering theorem to derive some very useful results. Here are two lemmas that will allow us to construct useful sets in the proof of the Nagata-Smirnov metrization theorem.

Lemma 5.2. Let $\mathcal{A}$ be a locally finite collection of subsets of X. Then

$$\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \bar{A}$$

Proof. Let $Y = \bigcup_{A \in \mathcal{A}} A$. In general, $\bigcup \bar{A} \subset \bar{Y}$ by basic properties of closure. We now aim to show the reverse inclusion using the assumption of local finiteness. Let $x \in \bar{Y}$. Let U be a neighborhood of x that intersects only finitely many element of $\mathcal{A}$, call them $A_1, ..., A_k$. Then x belongs to one of the sets $\bar{A}_1, ..., \bar{A}_k$, and thus $\bigcup \bar{A}$. If x belonged to none of these sets then $U - A_1 - ... - A_k$ would be a neighborhood of x that intersects no elements of $\mathcal{A}$ and thus would not intersect Y, contradicting the assumption that $x \in \bar{Y}$. □

Lemma 5.3. Let X be a metrizable space. If $\mathcal{A}$ is an open covering of X, then there is an open covering $\mathcal{S}$ of X refining $\mathcal{A}$ that is countably locally finite

Proof. Choose a well-ordering $<$ for the collection $\mathcal{A}$: such an ordering is guaranteed to exist by the well-ordering theorem. Similarly choose a metric for X which is guaranteed to exists since X is assumed to be metrizable. We will denote the elements of $\mathcal{A}$ by the letters $U, V, W, ...$

First, we construct a locally finite refinement of $\mathcal{A}$; We don't yet worry that the subsets are open or that they form a covering. Let n denote a positive integer, fixed for the moment. Given an element $U \in \mathcal{A}$ we define $S_n(U)$ to be the subset of U that is at least a distance $1/n$ within U. More precisely, let

$$S_n(U) = \{x | B(x, 1/n) \subset U\}$$

Now we use the well-ordering $<$ of $\mathcal{A}$ to further restrict the sets so that they don't overlap. For each $U \in \mathcal{A}$ let

$$T_n(U) = S_n(U) - \bigcup_{U<V} V$$

this sets are disjoint as the figure below suggests. Moreover they are separated by a distance of at least $1/n$. To see this, assume without loss of generality that $V < W$ with $x \in T_n(V)$ and $y \in T_n(W)$. Then by definition of $T_n(V)$ we also have $x \in S_n(V)$ and the $1/n$ neighborhood of x lies entirely in V. On the other hand $y \in T_n(W)$ implies by definition that $y \notin V$ which implies $y \notin B(x, 1/n)$ since the $1/n$ neighborhood of y lies entirely in V. This implies $d(x, y) \geq 1/n$.

Next, we'll expand the constructed sets $T_n(U)$ a bit so that we can be certain they're open sets. We'll expand them to an open set $E_n(U)$ where $E_n(U)$ is the $1/3n$ neighborhood of $T_n(U)$. More precisely

$$E_n(U) = \bigcup_{x \in T_n(U)} B(x, 1/3n)$$

Moreover, for any U and V distinct we have that $d(x, y) \geq 1/3n$ for any $x \in E_n(U)$ and $y \in E_n(V)$ by the triangle inequality so sets are disjoint. More explicitly, there must also exist $x' \in T_n(U)$ and $y' \in T_n(V)$ by definition. Then

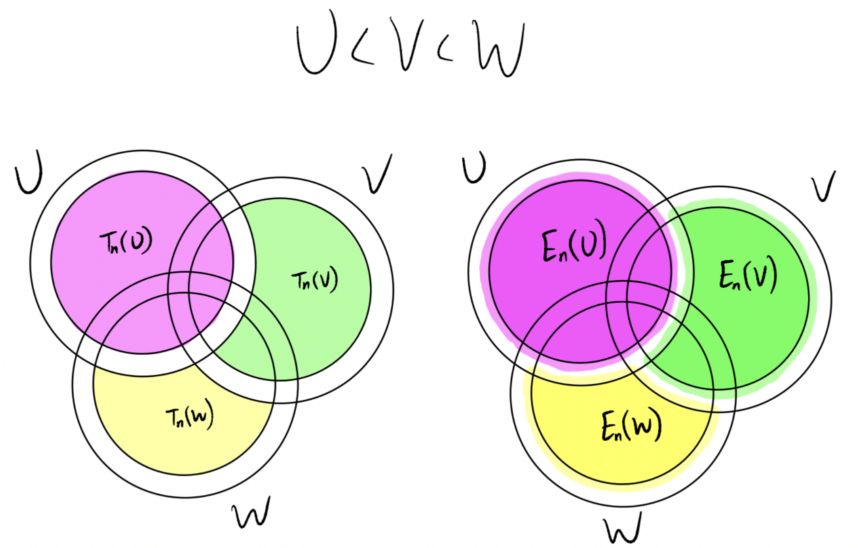

Figure 6: Locally Finite Open Cover

by the triangle inequality we have $1/n \leq d(x', y') \leq d(x', x) + d(x, y) + d(y, y') \leq 1/3n + d(x, y) + 1/3n$ which implies $d(x, y) \geq 1/3n$. See figure 6.

Now define

$$\mathcal{E}_n = \{E_n(U) | U \in \mathcal{A}\}$$

We claim that $\mathcal{E}_n$ a locally finite collection of open sets that refine $\mathcal{A}$. $\mathcal{E}_n$ refines $\mathcal{A}$ because $\mathcal{E}_n(U) \subset U$ for each $U \in \mathcal{A}$. $\mathcal{E}_n$ is locally finite because for any $x \in X$, the $1/6n$ neighborhood of x intersects at most one element of $\mathcal{E}_n$ because any two elements of $\mathcal{E}_n$ are separated by a distance of at least $1/3n$.

Of course, $\mathcal{E}_n$ does not cover X. However, we assert the collection

$$\mathcal{E} = \bigcup_{n \in \mathbb{Z}_+} \mathcal{E}_n$$

does cover X. To show this let x be a point of X. The collection $\mathcal{A}$ covers X by assumption. Then choose U to be the first element of $\mathcal{A}$ (in the well-ordering $<$) that contains x. Because U is open we can choose n so that $B(x, 1/n) \subset U$. Then, by definition, $x \in S_n(U)$. Moreover, because U is the first element of $\mathcal{A}$ that contains x, the point x belongs to $T_n(U)$ and thus also $E_n(U)$ which is an element of $\mathcal{E}_n \subset \mathcal{E}$. So $\mathcal{E}$ is an open covering of X refining $\mathcal{A}$ that is countably locally finite. $\square$

6 Nagata-Smirnov Theorem

We're nearly done. I introduce the definition of a G_δ set before we prove two lemmas on our way to the final theorem. These sets are nice because we can construct them using countable intersections of open sets. This will prove extremely useful for constructing continuous functions.

Definition 6.1 (G_δ set)**.** A subset A of a space X is called a $\mathbf{G}_\delta$ **set** in X if it equals the intersection of a countable collection of open subsets of X.

Example 10. In a metric space X, each closed set is a G_δ set. Given $A \subset X$, let $U(A, \epsilon)$ denote the ϵ-neighborhood of A. If A is closed then we have

$$A = \bigcap_{n \in \mathbb{Z}_+} U(A, 1/n)$$

so A is a G_δ set.

Lemma 6.1. Let X be a regular space with a basis $\mathcal{B}$ that is countably locally finite. Then X is normal, and every closed set in X is a G_δ set in X.

Proof. **Step 1.** Given W open in X we show that there is a countable collection $\{U_n\}$ of open sets of X such that

$$W = \bigcup U_n = \bigcup \bar{U}_n$$

Since the basis $\mathcal{B}$ is countably locally finite we can write $\mathcal{B} = \bigcup \mathcal{B}_n$ where each collection $\mathcal{B}_n$ is locally finite. Let $\mathcal{C}_n$ denote the set of basis elements $B \in \mathcal{B}_n$ such that $\bar{B} \subset W$. Then $\mathcal{C}_n$ is locally finite since each $\mathcal{C}_n$ is a subset of $\mathcal{B}_n$. Now define

$$U_n = \bigcup_{B \in \mathcal{C}_n} B$$

which is open as a union of open sets and by lemma 5.2 we also have

$$\bar{U}_n = \bigcup_{B \in \mathcal{C}_n} \bar{B}$$

so $\bar{U}_n \subset W$ because each $\bar{B} \subset W$ and further this implies that

$$\bigcup U_n \subset \bigcup \bar{U}_n \subset W$$

We assert that the equality holds. Given $x \in W$ there exist disjoint open sets U and V containing x and $X - W$ respectively since X is regular. Then there exists a basis element $B \subset U$ with $x \in B$ and $\bar{B} \subset W$ because $\bar{B} \subset \bar{U}$ and $\bar{U} \cap (X - W) = \varnothing$ since U and V are disjoint open sets and $X - W \subset V$. This

$B \in \mathcal{B}_n$ for some n. Then $B \in \mathcal{C}_n$ by definition and so $x \in U_n$. Then $W \subset \bigcup U_n$ and the equality follows.

Step 2. We show that every closed set C in X is a G_δ set in X. Given C, let $W = X - C$. By step 1, there are sets U_n in X such that $W = \bigcup \bar{U}_n$ and thus we have

$$C = X - W = X - \bigcup \bar{U}_n = \bigcap (X - \bar{U}_n)$$

Then C equals a countable intersection of open sets and is thus a G_δ set.

Step 3. We show that X is normal. Let C and D be disjoint closed sets By step 1 we construct a countable collection $\{U_n\}$ of open sets such that $\bigcup U_n = \bigcup \bar{U}_n = X - D$. Then $\{U_n\}$ covers C and each set $\bar{U}_n$ is disjoint from D. Similarly, there is a countable covering $\{V_n\}$ of D by open sets such that each $\bar{V}_n$ is disjoint from C. Then the sets $U = \bigcup U_n$ and $V = \bigcup V_n$ are open sets containing C and D respectively but they are not necessarily disjoint.

We perform a little trick to construct to open sets that *are* disjoint. Given n, define

$$U_n' = U_n - \bigcup_{i=1}^{n} \bar{V}_i \text{ and } V_n' = V_n - \bigcup_{i=1}^{n} \bar{U}_i$$

Each set U_n' is open as the difference of an open set U_n and a closed set $\bigcup_{i=1}^{n} \bar{V}_i$. The same logic shows that V_n' is open. The collection $\{U_n'\}$ covers A because each $x \in A$ belongs to some U_n but x does not belong to any $\bar{V}_i$ since each $\bar{V}_i$ is disjoint from C. Again, the same logic shows that the collection $\{V_n'\}$ covers D.

Now let

$$U' = \bigcup_{n \in \mathbb{Z}_+} U_n' \text{ and } V' = \bigcup_{n \in \mathbb{Z}_+} V_n'$$

Then U' is open as a union of open sets and contains C since the collection$\{U_n'\}$ covers C. Similar logic shows V' is open and covers D. We assert further that the two sets are disjoint. Suppose, for contradiction, that $x \in U' \cap V'$. Then $x \in U_j' \cap V_k'$ for some j and k. If $j \leq k$ then $x \in U_j$ by the definition of U_j' but by the definition of V_k' we have $x \notin U_j$. Similarly, if $k \leq j$ then $x \in V_k$ by the definition of V_k' but by the definition of U_j' we have $x \notin V_k$. In either case we have a contradiction, so U' and V' are distinct. Then we have disjoint open sets U' and V' containing C and D respectively, so X is normal. See figure 7. □

Lemma 6.2. Let X be normal and let A be a closed G_δ set in X. Then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$

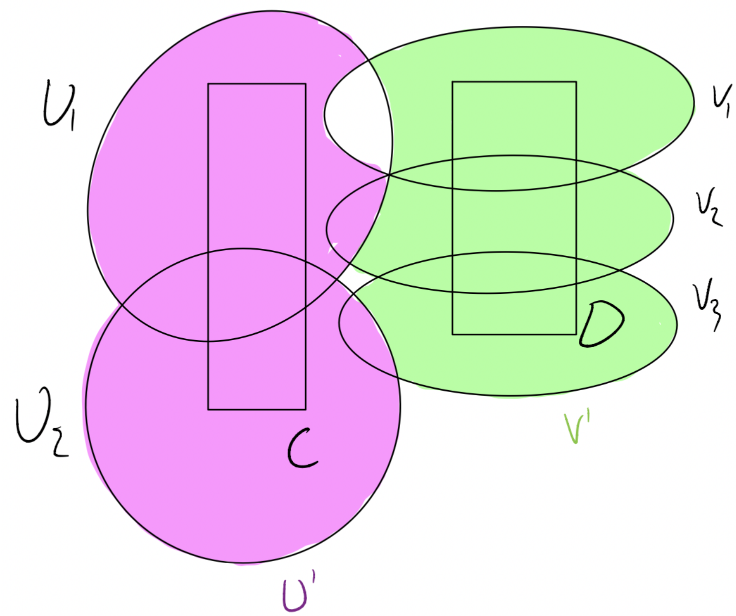

Figure 7: Disjoint Open Sets

Proof. Since A is a G_δ set let $A = \bigcap U_n$ where each U_n is open. Then $X - U_n$ is closed and the closed sets A and $X - U_n$ are disjoint because $A \subset U_n$ by definition. Then by the Urysohn lemma, there exists a continuous function $f_n : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in X - U_n$. Let $f(x) = \sum f_n(x)/2^n$. This series converges uniformly (compare it to the series $\sum 1/2^n$) and so f is continuous by the uniform limit theorem (see theorem 26.1 in Munkres for a proof). $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$ so this is the desired function. □

Theorem 6.3 (Nagata-Smirnov Metrization Theorem)**.** A space X is metrizable if and only if X is regular and has a basis that is countably locally finite

Proof. **Step 1.** Assume first that X is regular and has a countably locally finite basis $\mathcal{B}$. We will show that X is metrizable by embedding X in the metric space $(\mathbb{R}^J, \bar{\rho})$ for some J. We already know X is normal and every closed set in X is a G_δ set by lemma 6.1.

Let $\mathcal{B} = \bigcup \mathcal{B}_n$ where each $\mathcal{B}_n$ is locally finite. For each positive integer n

and each basis element $B \in \mathcal{B}_n$, choose a continuous function

$$f_{n,b} : X \to [0, 1/n]$$

such that $f_{n,b}(x) > 0$ for $x \in B$ and $f_{n,b} = 0$ for $x \notin B$. Such a function is guaranteed to exist by lemma 6.2 since $X - B$ is closed as the complement of a basis element, and thus a G_δ set. Observe that the collection $\{f_n, B\}$ separates points from closed sets in X: Given a point x_0 and a neighborhood U of x_0, there is a basis element B such that $x_0 \in B \subset U$. Then $B \in \mathcal{B}_n$ for some positive integer n so we have $f_{n,B}(x_0) > 0$ and $f_{n,B}(x) = 0$ for $x \in X - U \subset X - B$.

Let J denote the subset of $\mathbb{Z}_+ \times \mathcal{B}$ consisting of all pairs (n, B) where B is an element of $\mathcal{B}_n$. Define $F : X \to [0, 1]^J$ by the equation

$$F(x) = (f_{n,b}(x))_{(n,B)\in J}$$

and observe F is an embedding relative to the product topology on $[0, 1]^J$ by the embedding theorem.

Now we show that F is an embedding relative to the uniform topology on $[0, 1]^J$ as well. Clearly, F is still injective. Now recall that the uniform topology on $[0, 1]^J$ is finer than the product topology (see theorem 20.4 in Munkres for a proof). Then F is open relative to the uniform topology. For any U open in X, $F(U)$ is open in the product topology since F is an embedding, and thus $F(U)$ is open in the uniform topology since the uniform topology is finer than the product topology. We now need to prove that F is continuous.

Note that on the subspace $[0, 1]^J$ of $\mathbb{R}^J$, the uniform metric equals the metric $\rho(\mathbf{x}, \mathbf{y}) = \sup\{|x_\alpha - y_\alpha| : \alpha \in J\}$. To prove continuity, we take a point $x_0 \in X$ and a number $\epsilon > 0$, and find a neighborhood W of x_0 such that

$$x \in W \implies \rho(F(x), F(x_0)) < \epsilon$$

Fix n for the moment. Choose a neighborhood U_n of x_0 that intersects only finitely many elements of the collection $\mathcal{B}_n$. This is possible because $\mathcal{B}_n$ is locally finite. Then we have $f_{n,B}(U_n) = \{0\}$ for all but finitely many functions since only finitely many basis elements intersect U_n. For each of these finitely many remaining functions, because they are continuous, there exists a neighborhood around x_0 where $f_{n,B}$ varies from $f(x_0)$ by no more than $\epsilon/2$. Let V_n denote the intersection of these neighborhoods.

Choose such a neighborhood V_n of x_0 for each $n \in \mathbb{Z}_+$ and choose N sufficiently large so that $1/N \leq \epsilon/2$. Define $W = V_1 \cap ... \cap V_N$. We assert that W

is the desired neighborhood of x_0. Let $x \in W$. Then

$$n \leq N \implies |f_{n,B}(x) - f_{n,B}(x_0)| \leq \epsilon/2$$

because the function $f_{n,B}$ is either equal to zero on all of W or varies by at most $\epsilon/2$. Additionally,

$$n > N \implies |f_{n,B}(x) - f_{n,B}(x_0)| \leq 1/n < \epsilon/2$$

because $f_{n,B}$ maps x into $[0, 1/n]$. Therefore we have

$$\rho(F(x), F(x_0)) \leq \epsilon/2 < \epsilon)$$

as desired so F is continuous. F is injective, open, and continuous, so it can be embedded in the metric space $(\mathbb{R}^J, \bar{\rho})$. Then X is metrizable.

Step 2. Now we prove the converse. Assume X is metrizable. Then X is normal by theorem 2.3 and thus regular. Now we need to show that X has a countably locally finite basis.

Choose a metric for X. Given m let $\mathcal{A}_m$ denote the covering of X by all open balls of radius $1/m$. By lemma 5.2 there is an open covering $\mathcal{B}_m$ of X refining $\mathcal{A}_m$ that is countably locally finite. Observe that each element of $\mathcal{B}_m$ has diameter of at most $2/m$. Let

$$\mathcal{B} = \bigcup_{m \in \mathbb{Z}_+} \mathcal{B}_m$$

which is a countable union of countable sets thus countable. So then $\mathcal{B}$ is countably locally finite as well.

Last we show that $\mathcal{B}$ is a basis for X. Given $x \in X$ and $\epsilon > 0$ we show there is a basis element B such that $x \in B \subset B(x, \epsilon)$. First choose m sufficiently large so that $1/m < \epsilon/2$. Then because $\mathcal{B}_m$ covers X we can choose an element $B \in \mathcal{B}_m$ that contains x. Since B contains x and has diameter at most $2/m < \epsilon$, it is contained in $B(x, \epsilon)$ as desired. Since $x \in X$ and $\epsilon > 0$ were arbitrary, $\mathcal{B}$ is a basis. $\square$