Zariski Topologies and Sober Topological Spaces

David Chen, dc3451

December 22, 2021

Abstract

We explore sober topological spaces and in particular the Zariski topology, giving an overview on sobriety as a topological property before specializing into the Zariski topologies on varieties and commutative rings. The connection between prime ideals and algebraic varieties and the relationship is explored, giving geometric and algebraic meaning to the Zariski topology. The final section uses a result from Hochster's 1969 paper *Prime Ideal Structure in Commutative Rings* that a topological space is spectral if and only if it arises as the spectrum of a ring to unify the discussion in the first two sections. Knowledge of undergraduate ring theory is assumed.

Contents

Introduction

A central object of study in commutative algebra is the spectrum of a ring, defined as the set of all prime ideals of the ring. We denote the spectrum of some commutative ring A to be Spec(A), and can even topologize it by defining all the closed sets:

$$V(M) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid M \subset \mathfrak{p}\}$$

for every subset $M \subset A$. An interesting thing about this topology, called the Zariski topology, is that points aren't closed. Unlike many of the first examples of topology, we can have non-closed points: if we consider two prime ideals $\mathfrak{p} \subsetneq \mathfrak{q}$ then for any M, if $M \subset \mathfrak{p}$ then $M \subset \mathfrak{q}$ as well, so that any closed set in Spec(A) containing $\mathfrak{p}$ contains also $\mathfrak{q}$. Thus the singleton $\{\mathfrak{p}\}$ isn't closed, and in fact $\overline{\{\mathfrak{p}\}}$ contains every prime ideal which is a superset of $\mathfrak{p}$!

It is exactly in these types of topologies where the condition of sobriety becomes interesting. We'll first explore sober topological spaces in general before focusing on the Zariski topology, which will yield a really beautiful correspondence between algebra and geometry that lays the basis for the concepts of algebraic geometry. In particular, we will see the following:

Theorem. *There is a bijective correspondence between algebraic varieties of $\mathbb{A}_k^m$ and prime ideals in $k[x_1, \ldots, x_m]$.*

Roughly, this result tells you that varieties (sets where a collection of polynomials simultaneously vanish) correspond to the spectrum of the ring that the polynomials lie in, so that we can capture geometric information with algebraic properties!

At the end of the exposition, we'll discuss (but not prove) a result that unites the discussion between sobriety and the Zariski topology, namely,

Theorem (Hochster 1969)**.** *Let* **Spec** *be the subcategory of* **Top** *consisting of spectral spaces and maps. Then, there is a functor F :* **Spec** $\to$ **CRing** *such that the composition of* Spec *and F is isomorphic to the inclusion functor* **Spec** $\to$ **Top**.

As a direct corollary, this will give that every spectral space is the spectrum of some ring. This gives us a purely topological classification of the spectra of rings: we know exactly when a topological space arises as the spectrum of a commutative ring. Furthermore, it sort of justifies our specialization into Zariski topologies: every compact sober space with a basis of compact open sets arises as the Zariski topology on a commutative ring!

A note on convention before we properly begin. In a lot of algebraic literature, including Hochster's paper, compact is defined to imply Hausdorff. Then, if a space is not Hausdorff but still has each open cover admit a finite subcover, they call the space quasicompact. We will break from this tradition, and call spaces that always admit finite subcovers simply compact, not quasicompact. Next, all rings are assumed commutative with unity, and undergraduate-level ring theory is used freely without proof.

1 Sober Topological Spaces

1.1 Irreducibility

In the introduction, we have an example of a topology where the closure of a point is nontrivial. In many topological examples, this doesn't lead you anywhere interesting: in particular, if a space is Hausdorff, then the closure of a point is exactly itself. When when we do come across non-closed points in a topology, however, a natural thing to do would be to take its closure. Sobriety, then, is the condition that the "fundamental" closed subsets of a topological space are the closures of unique points. Let's make this precise!

Definition 1. A nonempty topological space X is **irreducible** if whenever we can write $X = E_1 \cup E_2$, both closed, either $X = E_1$ or $X = E_2$. Furthermore, a maximal irreducible subset $E \subset X$ is called an **irreducible component**.

These irreducible components are the "fundamental" closed subsets earlier; the justification and intuition for that signifier is that an irreducible components cannot be further partitioned into smaller closed subsets (note that this does not mean it cannot have closed subsets!) and so captures some fundamental piece of information about the topology.

We can further formally justify this intuition, and the nomenclature of "component" with the following properties:

Proposition 1. *Let X be a topological space.*

1. *The closure of an irreducible set is irreducible.*
2. *Irreducible components are closed.*
3. *Each irreducible subset lies in an irreducible component.*
4. *X is the union of its irreducible components.*

Proof. These are straightforward:

1. Let $E \subset X$ be an irreducible set. Then, if $\overline{E} = E_1 \cup E_2$, E_1, E_2 closed then $E = (E_1 \cap E) \cup (E_2 \cap E)$. Since E is irreducible, we can take WLOG $E = E_1 \cap E$. But E_1 is then a closed set containing E so that $\overline{E} \subset E_1$, so $\overline{E}$ is irreducible as well.
2. Since irreducible components are maximal, it must be that if E is an irreducible component, $E = \overline{E}$.
3. Let E be irreducible. Then, there is a partially ordered collection I of irreducible subsets $E \subset E_i \subset X$ via inclusion (that is, $i_1 \leq i_2 \iff E_{i_1} \subset E_{i_2}$). Let I' be a maximal totally ordered subset of I. Then, suppose $E' = \bigcup_{i \in I'} E_i = Z_1 \cup Z_2$, Z_1, Z_2 closed. Then, for each i, either $T_i \subset Z_1$ or $T_i \subset Z_2$; WLOG let there be some i_0 such that $T_{i_0} \subset Z_1$; then, by the definition of the ordering we see that for all $i \in I'$, $T_i \in Z_1$, so E' must be irreducible. We win by the maximality of I'.

4. Take $x \in X$. Then, $\{x\} \subset X$ is trivially irreducible, and by property 2 lives in some irreducible component.

□

At the start of this section, we mentioned that sobriety is uninteresting in Hausdorff spaces. We have the machinery to formally demonstrate this idea:

Proposition 2. *The irreducible components in a Hausdorff space X are exactly the singleton points.*

Proof. Note that if $E \subset X$ is irreducible and has two distinct elements $x, y \in X$, there are two disjoint sets U, V separating x and y. However, if $U, V \subset E$ open and $U \cap V = \emptyset$, then $E = (U \cap V)^c = U^c \cup V^c \implies U^c = E$ or $V^c = E$ by irreducibility so one of U, V must be empty. This means that E cannot be Hausdorff, so all irreducible components must be singletons. But singletons are trivially irreducible, so we get our original observation. □

1.2 Sobriety

Definition 2. We now name the combination of two facts: first, that points are irreducible, and that closures of irreducible sets are irreducible. In particular,

1. Suppose $E \subset X$ is irreducible and closed, and $x \in E$ satisfies $\overline{\{x\}} = E$. Then, x is called a **generic point** of E.

2. A space X is called **sober** if every irreducible closed subset of X has a generic point.

We can see now that Hausdorff spaces are sober: all irreducible closed subsets of a Hausdorff space are points, and are their own closures. To get some more intuition, we will make a couple more definitions before presenting examples.

Definition 3. If we have points $x, y \in X$ and $x \in \overline{y}$, we say that x is a **specialization** of y or y is a **generalization** of x (hence, in the definition of generic point, x is a generalization of everything in E).

Let's work with some visual examples: the following diagrams will have irreducible subsets circled with irreducible components circled in red and specializations marked with arrows.

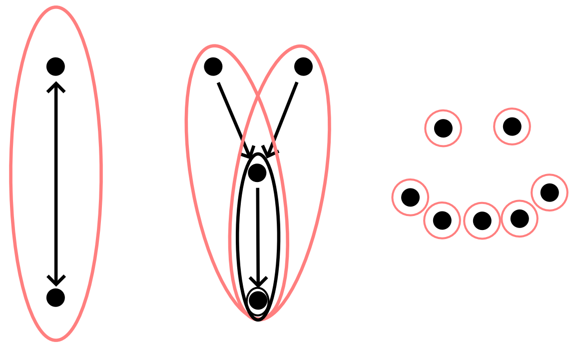


Figure 1: Left: the indiscrete topology on two elements. Middle: A sober, non-Hausdorff space; the black circle is a irreducible set which is not an irreducible component. Right: A Hausdorff space.

The left is actually the indiscrete topology on two elements. This is not sober, since the unique irreducible component is the set with both points, which has two generic points. We can work this out explicitly to make the diagram more clear: let $X = \{a, b\}$ with the indiscrete topology. Then, there is necessarily only a single irreducible component (since there is only one nonempty closed set), namely $\{a, b\}$. Then, we can see that $\overline{\{a\}} = \overline{\{b\}} = \{a, b\}$, and in particular a specializes to b and b specializes to a; this is reflected by the double sided arrow between the two points representing a and b.

The middle is an example of a sober space: we see that the two irreducible components have unique generic points, namely the two points at the top of the diagram. Visually, we can really see the fact that this is irreducible because in each irreducible component (circled in red) we can go from any point and trace the specializations backwards to a single unique point, which will be the desired generic point. The rightmost figure is a Hausdorff space, where there are no nontrivial specializations and the diagram is sort of flat: the irreducible components are just points.

A natural question, once we've felt out the initial definitions of sobriety, might be if it is equivalent to some other more familiar topological property. We've already seen that T_2 directly implies sober, and it is easy to check that every sober space is T_0:

Proposition 3. *If X is sober, it is T_0, also known as Kolmogorov.*

Proof. Suppose that $x, y \in X$ were points where every closed set contains either both or none of them. In that case, we have that every closed set containing x and y must contain either both or neither, such that $\overline{\{x\}} = \overline{\{y\}}$. □

This might give us hope at simplifying the situation: after all we know that both

$$T_2 \implies T_1 \implies T_0$$
$$T_2 \implies \text{sober} \implies T_0$$

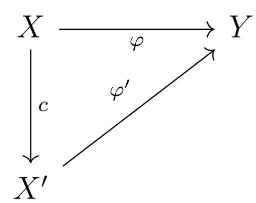
so perhaps spaces are T_1 if and only if they are sober! But this unfortunately isn't true; $\mathrm{Spec}(\mathbb{C}[x])$ will be a counterexample: note that $(0) \subset (x)$ so that $\{(0)\}$ is not a closed set, and all points are closed in T_1 spaces. We'll expand more on spaces like this in the next section.

Here is a nice result about a universal property related to sober spaces that will be used later in showing the Zariski topology is sober.

Theorem 1. *(Soberification) Let X be a topological space. Then, there is an associated topological space X' and a canonical continuous map*

$$c : X \to X'$$

which is universal among maps to sober spaces. That is, if Y is any sober topological space this diagram commutes:



Proof. Consider X' to be the set of all irreducible subsets of X and the mapping $x \mapsto \overline{x}$, where the topology on X' is defined via the following construction: for every open set $U \subset X$, let U' be the set of irreducible closed subsets that meet U. Then, the open sets of X' will be all the U', and in particular if $U_1 \neq U_2$, then $U_1' \neq U_2'$ since $x \in U_1, x \notin U_2 \implies \overline{\{x\}} \in U_1', \overline{\{x\}} \notin U_2'$. Thus, there is a bijection between opens of X and opens of X'.

Now take any $U \subset X$ open. Certainly for any $x \in U$, $\overline{\{x\}} \in U'$, so that $x \in c^{-1}(U') \implies U \subset c^{-1}(U')$. Then, if $y \in \overline{x} \cap U$ if $y = x$ then $x \in U$; otherwise y is a limit point of x and so every neighborhood of y contains x, so in particular U contains x. In either case, $x \in U$ so that $c^{-1}(U') \subset U$ as well.

But since $c^{-1}(U') = U$ by construction, our mapping is continuous, and furthermore $U' \subset X'$ is open if and only if $U = c^{-1}(U')$ is open since the open sets of X' are exactly the U'. Additionally, $U \mapsto U'$ respects unions and intersections: $U_1' \cap U_2' = (U_1 \cap U_2)'$ and $\bigcup_{\alpha \in I} U_\alpha' = \left(\bigcup_{\alpha \in I} U_\alpha\right)'$ (we omit the uninteresting details of this last statement for brevity), so we can see that this in fact does form a topology.

To see that X' is sober, let $T \subset X'$ be an irreducible closed subset. Then, there is some open $U \subset X$ such that $X' \setminus T = U'$; now consider $Z = X \setminus U$. In particular, if $Z = Z_1 \cup Z_2$, both closed proper subsets of Z, we have that since $U \mapsto U'$ respects unions,

$$U = Z^c \implies U' = X' \setminus T = T^c = (Z^c)' = (Z_1^c \cap Z_2^c)' = (Z_1^c)' \cap (Z_2^c)'$$

If we now take complements, we see that T is then the union of two closed sets, since $(Z_1^c)', (Z_1^c)'$ are open sets in X'. This contradicts irreducibility of T, as neither $(Z_1^c)', (Z_1^c)'$

can be all of T^c without violating the bijectivity of $U \mapsto U'$. Thus, we can see that Z is irreducible.

Now, we claim that Z is the unique generic point of T; this is because any open of the form $V' \subset X'$ not containing Z must originate from an open $V \subset X$ which misses Z from our definition of V' as the collection of irreducible closed subsets that meet V. Then, $V \subset U \implies V' \subset U'$, since any irreducible closed subset that meets V certainly must also meet U. Now, if $Z \in S \subsetneq T$ with S closed, then $S^c \subset U = T^c$; but $S \subsetneq T \implies S^c \supsetneq T^c$, so there cannot be such an S and thus T is the smallest closed subset containing Z.

The above shows that $\overline{\{Z\}} = T$; to show uniqueness, suppose $\overline{\{Y\}} = T$. The above paragraph with $S = \overline{\{Y\}} = \overline{\{Z\}} = T$ shows that $\overline{\{Y\}}^c = V \subset U$ and $\overline{\{Z\}}^c = U \subset V$, so in fact $Z = Y$. So we see that every irreducible closed $T \subset X'$ has a generic point.

We now demonstrate the universal property. Let $\varphi : X \to Y$ be a continuous map and Y a sober space. Let $\varphi' : X' \to Y$ be the map sending some irreducible closed $Z \subset X$ to the unique generic point of $\overline{\varphi(Z)}$. It is a general topological fact that φ is continuous if and only if for every $E \subset X$, $\varphi(\overline{E}) \subset \overline{\varphi(E)}$. Then, $\overline{\varphi(\overline{\{x\}})} \subset \overline{\varphi(x)}$; but at the same time, $\overline{\varphi(x)} \subset \overline{\varphi(\overline{\{x\}})}$, so it must be that $\varphi(x)$ is a generic point of $\overline{\varphi(\overline{\{x\}})}$, so $(\varphi' \circ c)(x) = \varphi(x)$, which is what we wanted.

For continuity, if $V \subset Y$ is open, $\varphi^{-1}(Y)$ is open $\implies c^{-1}(\varphi'^{-1}(V))$ is open; but since $c^{-1}(U')$ is open if and only if U is open, $\varphi'^{-1}(V)$ must be open, so φ' is continuous. This concludes our proof of the universality of X'. □

Corollary 1. The above map is a bijection if and only if X is sober.

Proof. Suppose $c : X \to X'$ is bijective. Since c is a surjection, every irreducible closed subset is some $c(x) = \overline{\{x\}}$ and thus the closure of a point, so each irreducible closed subset has a generic point. Further, since c is injective, this generic point is unique (otherwise, if Z has two distinct generic points x and y, we would have $Z = c(x) = c(y)$).

On the other hand, if X is sober, then the identity map $\mathrm{id} : X \to X$ factors into $\mathrm{id}' \circ c$ by the universal property. However, by the construction in the proof, id' must take $\overline{\{x\}} \mapsto x$, so c is invertible and thus a bijection. □

1.3 Dimension

Krull dimension, or combinatorical dimension, is a very different notion from the usual definition we see in something like analysis (but there is a geometric connection, which we will see later!). In particular, it is defined relative to the irreducible subsets of the topology as follows:

Definition 4. Let X be a topological space. Then,

1. A **chain** of irreducible closed subsets of length n is a sequence $E_0 \subsetneq E_1 \subsetneq \ldots, \subsetneq E_n \subset X$ with E_i irreducible closed.

2. The **Krull dimension** of X is the supremum of the length of all irreducible closed chains, and is allowed to be infinite.

To see that this diverges from our usual notion of dimension, note that $\mathbb{R}^n$ is Hausdorff, so all irreducible subsets are points and thus there are no proper inclusions of irreducible subsets. Thus, $\mathbb{R}^n$ with the usual topology is of Krull dimension 0; this is why the diagram in Figure 1 looks so "flat": all the points are lying in dimension 0.

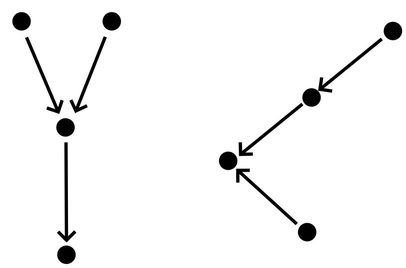


Figure 2

In the figure above, the arrows are specializations, and the points are the only points in the topology. The figure on the left has dimension 2 and has equidimensional chains; the figure on the right also has dimension 2 but not equidimensional chains. Both are also sober.

For an example with infinite dimension take a rather silly topology: $\mathbb{N}$ with closed sets $E_n = \{0, 1, \ldots, n\}$ for any n. Arbitrary intersections still get you closed sets, as do finite unions, so this is well defined. In this case, the unique irreducible component is $\mathbb{N}$ itself, and every point in $\mathbb{N}$ becomes a generic point of this component. Further, this clearly has infinite dimension since $E_1 \subset E_1 \subset \ldots$ is a chain of irreducible closed subsets.

2 The Zariski Topology

We are now prepared to move on to a more specific discussion of the Zariski topology in particular. We will cover two versions: one on (affine) algebraic varieties, and the other on the spectra of commutative rings. The correspondence between the two is a fundamental connection between algebra and geometry!

2.1 On Affine Algebraic Varieties

For the rest of this section, k is an algebraically closed field. If you like, you can take $k = \mathbb{C}$ not be mislead about anything here.

Definition 5. Take $f_1, f_2, \ldots f_n \in k[x_1, \ldots, x_m]$. An (affine) **algebraic set** is defined to be, for any selection of the polynomials before, the vanishing locus of all the polynomials:

$$Z(f_1, \ldots, f_n) = \{a = (a_1, \ldots, a_m) \in k^m \mid f_1(a) = \cdots = f_n(a) = 0\}.$$

An (affine) **algebraic variety** is an affine algebraic set which is irreducible: it cannot be written as the union of two smaller affine algebraic sets.

Now, we can already potentially begin to see where the earlier leadup is taking us: this irreducibility criterion sounds a lot like the claim for irreducible sets! In fact, we can topologize affine sets to make this formal:

Definition 6. The **Zariski topology** on $\mathbb{A}_k^m$ (m-tuples of elements of k is defined via closed sets, very similar to how we introduced the Zariski topology in the introduction on prime ideals. In particular, the closed sets will be exactly the algebraic sets.

In particular, the earlier formulation creates a topology on the affine plane $\mathbb{A}_k^m$, and the Zariski topology on any variety is simply the inherited subspace topology. We want to check that this in fact defines a topology.

Lemma 1. *We have the following properties for the algebraic sets on $\mathbb{A}_k^m$:*

1. $Z(f_1, \ldots, f_n) = Z((f_1, \ldots, f_n))$, *where $(f_1, \ldots, f_n)$ is the ideal generated by all the $f_1, \ldots, f_n$.*
2. $Z(I) \cup Z(J) = Z(IJ)$.
3. $\bigcap_{\alpha \in A} Z(I_\alpha) = Z\left(\sum_{\alpha \in A} I_\alpha\right)$, *where $\sum_{\alpha \in A} I_\alpha$ is the ideal generated by finite combinations of elements in I_α.*

Proof. We proceed in order; all of the following are pretty basic set theory.

1. Clearly $Z((f_1, \ldots, f_n)) \subset Z(f_1, \ldots, f_n)$. Suppose that $g = \sum_{i=1}^n c_i f_1$; then, clearly if $f_1(a), \ldots, f_n(a) = 0$ for all i, then $g(a) = 0$ as well, so every polynomial in the ideal generated by $f_1, \ldots, f_n$ also vanishes on $Z(f_1, \ldots, f_n)$, so $Z(f_1, \ldots, f_n) \subset Z((f_1, \ldots, f_n))$.
2. Suppose that $a \in Z(IJ)$; then $fg \in IJ$ vanishes at a for any $f \in I, g \in J$. Then, since k is a field (and thus an integral domain) we have that either $f(a) = 0$ or $g(a) = 0$. Then $a \in Z(I) \cup Z(J)$, and so $Z(IJ) \subset Z(I) \cup Z(J)$.

   On the other hand, if $a \in Z(I) \cup Z(J)$, for any $fg \in IJ$ with $f \in I, g \in J$, either $f(a) = 0$ or $g(a) = 0$, or both, so $a \in Z(IJ)$ and $Z(I) \cup Z(J) \subset Z(IJ)$.
3. Suppose that $a \in \bigcap_{\alpha \in A} Z(I_\alpha)$. Then, certainly every finite linear combination of polynomials in I_α vanishes at a, since each individual term is killed at a. Similarly, if $a \in Z(\sum_{a \in A} I_\alpha)$, then as a special case for every $\alpha \in A$, $a \in Z(I_\alpha) \implies a \in \bigcap_{\alpha \in A} Z(I_\alpha)$.

□

Let's draw some examples of affine varieties and see what specializations and irreducible subsets look like. As an example, consider $\mathbb{A}_k^2$, which will be drawn as $\mathbb{R}^2$, with quadratics being drawn as parabolas, linear equations as lines, etc.

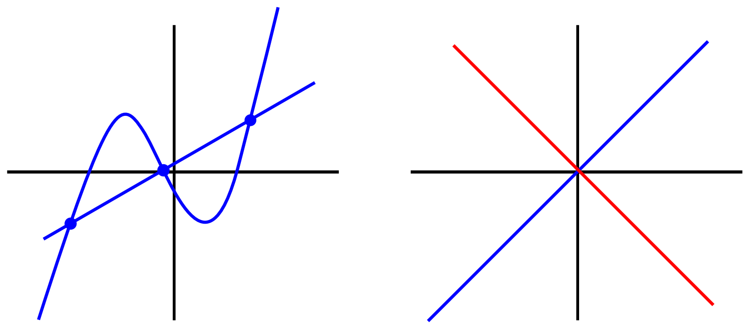


Figure 3: Left: two irreducible varieties (in blue, with black axes) which intersect to a reducible variety. Right: $Z(x^2 - y^2)$, with the irreducible subvarieties in different colors.

The left figure shows two varieties: one of the form $Z(g)$ where $g(x, y) = y - x^3 - ax - b$ and another where $Z(f)$ where $f(x, y) = y - ax - b$; depending on the field, this picture may or may not reflect the full nature at hand, but gives a good heuristic to think about varieties. The two graphs of the functions are irreducible subsets in $\mathbb{A}_k^2$. Next, we consider not $Z(f)$ and $Z(g)$, but $Z(f, g)$ instead, which consists of a few points in the plane. This isn't irreducible, since it is the union of a finite amount of algebraic sets consisting of only single points.

As a precursor to our next section, an algebraic property of some polynomial f can be translated into a topological statement on $\mathbb{A}_k^m$. Suppose that $f(x, y) = f_1(x, y)f_2(x, y)$; then it is true that algebraically $(f) = (f_1)(f_2)$ as a product of ideals, so from earlier $Z(f) = Z(f_1) \cup Z(f_2)$, and in particular the right picture is an example: the two different colors are the two irreducible varieties corresponding to $Z(x - y)$ and $Z(x + y)$.

2.2 As the Spectrum of a Ring

The above examples make it very clear that varieties are intimately tied to polynomials algebras over the base fields - but these are rings! This connection will be made more formal in the following section.

We've already seen the definition for the spectrum of a ring in the introduction; we restate the definition here.

Definition 7. The spectrum of a ring A, denoted $\mathrm{Spec}(A)$, is the set of all prime ideals of A. The Zariski topology on $\mathrm{Spec}(A)$ is given by specifying the closed sets to be, for any $M \subset A$,

$$V(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M \subset \mathfrak{p}\}$$

We will present without proof the following facts that ensure that $\text{Spec}(A)$ has a well-defined topology.

Proposition 4. *For I, J ideals in some ring A,*

1. *$V(M) = V(I)$, where M is the ideal generated by M.*
2. *$V(I) \cup V(J) = V(IJ)$.*
3. *$\bigcap_{\alpha \in A} V(I_\alpha) = V\left(\sum_{\alpha \in A} I_\alpha\right)$.*

The reason the proofs are omitted is because this proposition is exactly the same as when we were dealing with the Zariski topology on affine varieties! We'll instead establish a connection between the two, and that will serve as intuition for why the above must actually hold.

Let's instead prove some properties about the spectrum of a ring. Not only do they give us insight into the topological structure of ring spectra, but they'll be used in the last section when discussing sobriety.

Lemma 2. *Let A be a ring, $\mathfrak{p} \in \text{Spec}(A)$. We have the following:*

1. *$V(M) = V(\sqrt{I})$, where I is the ideal generated by M, and $\sqrt{I}$ the radical of M.*
2. *$\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.*
3. *Maximal ideals are closed points.*
4. *Irreducible subsets are exactly of the form $V(\mathfrak{p})$.*
5. *If $\mathfrak{p}$ is minimal, then $V(\mathfrak{p})$ is an irreducible component.*

Proof. Again, we go in order.

1. Clearly $M \subset I \subset \sqrt{I} \implies V(I) \subset V(M)$; but if $M \subset \mathfrak{p}$ then since $\sqrt{I}$ is the intersection of all prime ideals containing M, $\sqrt{I} \subset \mathfrak{p}$ as well, so $V(M) \subset V(\sqrt{I})$.

2. Suppose $\mathfrak{p} \in V(I)$ for some ideal I. Then, $I \subset \mathfrak{p}$; now consider $\overline{\{\mathfrak{p}\}} = \bigcap_{I \subset \mathfrak{p}} V(I)$. However, it is clear that $I \subset \mathfrak{p} \implies V(I) \subset V(\mathfrak{p})$, so $\bigcap_{I \subset \mathfrak{p}} V(I) = V(\mathfrak{p})$.

3. Let $\mathfrak{m}$ be a maximal ideal; then $V(\mathfrak{m}) = \{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{m} \subset \mathfrak{p}\}$; but since $\mathfrak{m}$ is maximal, the only prime ideal containing it is itself, so $V(\mathfrak{m}) = \overline{\mathfrak{m}}$.

4. That $V(\mathfrak{p})$ is irreducible is implied by 2, and the fact that the closure of a point is irreducible. Now, consider $V(I)$ for some nonprime radical ideal I and choose $a, b \notin I$ such that $ab \in I$. Now if $\mathfrak{p} \in V(I)$, $ab \in \mathfrak{p} \implies a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ since $\mathfrak{p}$ is prime, so $\mathfrak{p} \in V(I, a) \cup V(I, b)$. Similarly, if $\mathfrak{p} \in V(I, a) \cup V(I, b)$, then $I \subset \mathfrak{p}$, so we can conclude $V(I) = V(I, a) \cup V(I, b)$.

   But if $V(I) = V(I, a)$, then I being radical means that $I = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p} = \bigcap_{\mathfrak{p} \in V(I)} \implies a \in I$. Since we took $a \notin I$, $V(I) \neq V(I, a)$; an identical argument shows $V(I) \neq V(I, b)$. Thus $V(I)$ is not irreducible.

5. This is directly implied by 4: since $\mathfrak{p} \subset \mathfrak{q} \implies V(\mathfrak{q}) \subset V(\mathfrak{p})$, the maximal irreducible subsets must be associated to minimal primes.

□

We can also give an explicit basis for the Zariski topology:

Lemma 3. *Let A be a ring. Put, for any element $f \in A$, $D(f) = \text{Spec}(A) \setminus V(f)$ as a basic open set. Then, $\{D(f) \mid f \in A\}$ form a basis for* Spec(A)*.*

Proof. We need to check that for any $\mathfrak{p} \in U$ open, there is some $f \in A$ such that $\mathfrak{p} \in D(f) \subset U$. In particular, since U is open, it is the complement of some closed set, and is thus $\text{Spec}(A) - V(I)$ for some ideal I. Then, we have that for $\mathfrak{p} \in U$, $\mathfrak{p} \notin V(I) \implies I \not\subset \mathfrak{p}$; then, there is some element $f \in I$ where $a \notin \mathfrak{p}$, in which case $\mathfrak{p} \notin D(f)$. Furthermore, we have that since $f \in I$, $V(I) \subset V(f) \implies \text{Spec}(A) - V(f) \subset \text{Spec}(A) - V(I)$ so $D(f) \subset U$ as well, so we have what we want.

Then, by taking $U = \text{Spec}(A)$, the basic open sets clearly cover $\text{Spec}(A)$. Similarly, by taking $U = D(f_1) \cap D(f_2)$, we can find some f such that $\mathfrak{p} \subset D(f) \subset D(f_1) \cap D(f_2)$, so $\{D(f) \mid f \in A\}$ for a basis for $\text{Spec}(A)$. □

As a first example, $\text{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ prime }\} \cup \{(0)\}$. In a more topical vein, however, we can ask: what is $\text{Spec}(k[x, y])$ for an algebraically closed field k? To answer this question and the natural generalization to $k[x_1, \ldots, x_m]$, I'm going to state and use the following algebraic result:

Theorem 2 (Hibert Nullstellensatz)**.** *Let k be an algebraically closed field, I an ideal in $A = k[x_1, \ldots, x_m]$, and let $f \in A$ be a polynomial which vanishes at all points of $Z(I)$. Then $f^r \in I$ for some positive integer r. Equivalently, for any algebraic set $Y \subset \mathbb{A}_k^m$, define*

$$I(Y) = \{f \in A \mid f(a) = 0, \forall a \in Y\}$$

Then, if $J \subset A$ is an ideal, $I(Z(J)) = \sqrt{J}$, where $Z(J)$ is the vanishing locus discussed in the earlier section on algebraic varieties.

Corollary 2. There is an inclusion reversing bijective correspondence between algebraic sets in $\mathbb{A}_k^m$ and radical ideals in $k[x_1, \ldots, x_m]$. Furthermore, this restricts to a bijective correspondence between prime ideals and algebraic varieties.

Proof. The first part is implied by the Nullstellensatz directly by considering the correspondence generated by $J \mapsto Z(J)$ and $Y \mapsto I(Y)$, which interchange algebraic sets and radical ideals. We omit the details in lieu of the second part, which is more topical to the subjects of this exposition.

Now let Y be an algebraic variety; if $fg \in I(Y)$, then we have $\forall a \in Y, (fg)(a) = 0 \implies Y \subset Z(fg) = Z(f) \cup Z(g)$. Thus, since $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$ and Y is irreducible, WLOG we must have that $Y = Y \cap Z(f)$, so that $Y \subset Z(f) \implies f \in I(Y)$ by the Nullstellensatz. But then, this directly shows that $I(Y)$ is prime.

Conversely, if $\mathfrak{p}$ is a prime ideal and $Z(\mathfrak{p}) = Y_1 \cup Y_2$, $\mathfrak{p} = I(Y_1) \cap I(Y_2)$ by the Nullstellensatz and the fact that $\mathfrak{p} = \sqrt{\mathfrak{p}}$. However, $I(Y_1)I(Y_2) \subset I(Y_1) \cap I(Y_2) = \mathfrak{p} \implies I(Y_1) = \mathfrak{p}$ or $I(Y_2) = \mathfrak{p}$ by the primality of $\mathfrak{p}$. But this directly implies that $I(\mathfrak{p}) = Y_1$ or Y_2. Thus, $Z(\mathfrak{p})$ for a prime $\mathfrak{p}$ is a variety, and $I(Y)$ for a variety Y is a prime. □

Corollary 3. A maximal ideal in $k[x_1, \ldots, x_m]$ must correspond to minimal irreducible closed subsets of $\mathbb{A}_k^m$, which from earlier examples must be a point in the affine plane, say $(a_1, \ldots, a_m)$. Then, we see that maximal ideals are of the form $\mathfrak{m} = (x_1 - a_1, \ldots, x_m - a_m)$.

More generally, we see that $\text{Spec}(k[x_1, \ldots, x_m])$ is really equivalent to the topology on the affine plane, as ; this correspondence is why both topologies are called the Zariski topology! However, the spectrum of a ring carries a little more information, as we'll see in the examples at the end of the section.

Now, if we want to examine the topology of a specific variety in the affine plane, we want to study its associated coordinate ring.

Definition 8. For some algebraic variety $Z(f_1, \ldots, f_n)$, its associated **coordinate ring** is $k[x_1, \ldots, x_m]/(f_1, \ldots, f_n)$.

Let's look at a couple of examples.

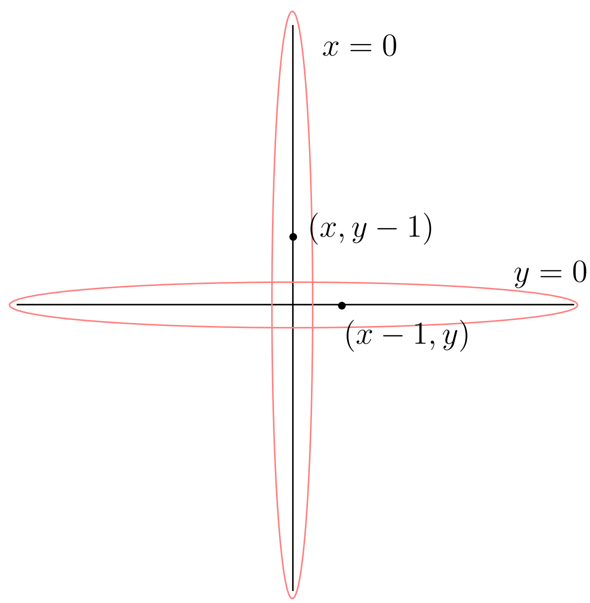


Figure 4: $\text{Spec}(k[x, y]/(xy))$, or alternatively the algebraic variety $Z(xy) = \{(x, y) \mid xy = 0\}$

This is a diagram of the spectrum of the coordinate ring of $\{(x, y) \mid xy = 0\}$, or $\text{Spec}(k[x, y]/(xy))$. This makes the geometric correspondence much clearer: for example, the maximal ideals are $(x, y - b)$ and $(x - a, y)$ corresponding to points on the cross $(0, b)$ and $(a, 0)$.

As in the earlier figures, the two nontrivial irreducible components are circled in red; inside the ring, they correspond to the prime ideals (x) and (y) (in the correspondence to algebraic varieties, they correspond to the set $\{(x, y) \in \mathbb{A}_k^2 \mid x = 0\}$ and $\{(x, y) \in \mathbb{A}_k^2 \mid y = 0\}$ respectively).

Furthermore, topological dimension makes its geometric manifestation here as well. The prime ideal structure shows that there are two smaller ideals (x) and (y) which are minimal, sitting below the maximal primes. In the dot diagrams of part 1,

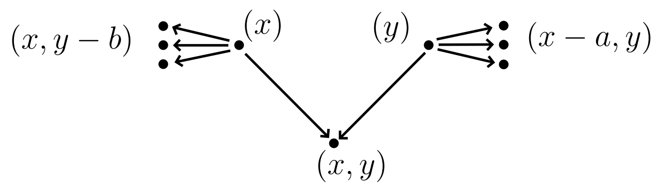


Figure 5: $(\mathrm{Spec}(k[x, y]/(xy)))$

We can see from this diagram that in some sense, the ideals $(x), (y)$ lie in dimension 0 (diagrammatically, no arrows point to them, so no smaller ideals specialize to them), and $(x, y - b), (x - a, y)$ lie in dimension 1; once we pass this through the inclusion reversing correspondence to algebraic varieties, we get that the varieties $\{(x, y) \mid x = 0\}$ and $\{(x, y) \mid y = 0\}$ lie in dimension 1 and the points lie in dimension 0: exactly the geometric intuition we originally expected!

We end off this section with a discussion on generic points and specializations. In the above example, once we pass to algebraic varieties, we see that specializations move us from "large" varieties (e.g. the axes of the cross) to "small" varieties (e.g. the points in the plane lying on the axes), so we are specializing from "large" curves to "smaller" curves! And the generic points of the spectra are exactly the largest (and therefore most "general") varieties, corresponding to the minimal ideals: the generic points of $\mathrm{Spec}(k[x, y]/(xy))$ are the lines corresponding to (x) and (y), the largest pieces of the affine plane and the smallest primes of the coordinate ring!

The key part of introducing Spec however, is that we can actually have this discussion at all: before, in the setup with varieties, there is no idea of specializations or generic points since the axes would be described by something like $Z(x)$; but x is a polynomial, not a point in $\mathbb{A}_k^2$ and the axes themselves were clearly not points in $\mathbb{A}_k^2$, so we couldn't really say that the axes were specializations of anything, or that the line is a generic point of the variety. However, inside the ring-theoretic setup, the axes are actually points in the spectrum!

3 Spectral Spaces

We now want to topologically characterize these structures which come up algebra and geometry via only topological considerations; the idea that $X \cong \mathrm{Spec}(A)$ if and only if X is spectral is exactly what we want, and we can even make this equivalence functorial.

This section will introduce the main terms and considerations of the result stated in the beginning. The full proof of the result is omitted; it is highly technical and relies on a *lot* of algebraic machinery. Instead, the goal of this exposition is to use the result to unify our discussion of the sobriety and the Zariski topology: sober spaces, with a couple of restrictions, are always homeomorphic to some Zariski topology!

3.1 Spec as a Functor

Recall that in the introduction, we not only used Spec to denote the spectrum of a ring, but also as a functor from **CRing**, the category of commutative rings, to **Top**, the category of topological spaces. Let's make this precise. First, we check that Spec takes ring maps to continuous maps:

Lemma 4. *Every ring map* $f : A \to B$ *induces a continuous map* $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *given by* $\varphi(\mathfrak{p}) = f^{-1}(\mathfrak{p})$.

Proof. We first check that $f^{-1}(\mathfrak{p})$ is prime: if $ab \in f^{-1}(\mathfrak{p})$, then $f(ab) = f(a)f(b) \in \mathfrak{p}$, so one of $f(a), f(b)$ must be in $\mathfrak{p}$, and thus one of a, b must be in $f^{-1}(\mathfrak{p})$.

It is now enough to show that $\varphi^{-1}(D(a))$ is open for any $a \in A$, since $D(a)$ form an open basis for $\mathrm{Spec}(A)$. But when we recall the definition of $D(a)$,

$$\mathfrak{p} \in \varphi^{-1}(D(a)) \iff \varphi(\mathfrak{p}) \in D(a) \iff a \notin \varphi(\mathfrak{p}) = f^{-1}(\mathfrak{p})$$

but now we have

$$a \notin f^{-1}(\mathfrak{p}) \iff f(a) \notin \mathfrak{p} \iff \mathfrak{p} \in D(f(a))$$

and since $D(f(a))$ is an open set in $\mathrm{Spec}(B)$, φ is continuous. □

We can now properly check that Spec is a functor, and we call the induced map in the earlier lemma $\varphi = \mathrm{Spec}(f)$.

Lemma 5. Spec *is a contravariant functor taking* **CRing** $\to$ **Top**.

Proof. Certainly if $f : A \to A$ is the identity, $\varphi = \mathrm{Spec}(f)$ must take $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p}) = \mathfrak{p}$, so Spec takes the identity to the identity.

Let $f : A \to B, g : B \to C$. We need to show that $\mathrm{Spec}(g \circ f) = \mathrm{Spec}(f) \circ \mathrm{Spec}(g)$. Put $\varphi = \mathrm{Spec}(g \circ f), \varphi_f = \mathrm{Spec}(f)$, and $\varphi_g = \mathrm{Spec}(g)$. Then,

$$\varphi(\mathfrak{p}) = (g \circ f)^{-1}(\mathfrak{p}) = (f^{-1} \circ g^{-1})(\mathfrak{p}) = f^{-1}(\varphi_g(\mathfrak{p})) = \varphi_f(\varphi_g(\mathfrak{p}))$$

so we have what we want. □

3.2 Spectral Spaces and Spectra of Rings

We are now ready to characterize the spectra of rings topologically.

Definition 9. If a topological space X satisfies the following, it is called **spectral**:

1. X is sober.
2. X is compact.
3. The compact subsets of X are closed under finite intersection and form an open basis of X.

Definition 10. Furthermore, if X, Y are spectral spaces, a mapping $X \to Y$ is called **spectral** if it is continuous and the preimage of any open and compact subset of Y is again compact.

Proposition 5. *Let A be a ring; then,* $\mathrm{Spec}(A)$ *is spectral. Further, if $f : A \to B$ is a ring map, the induced map on spectra* $\mathrm{Spec}(f)$ *is spectral.*

Proof. We will only check that $\mathrm{Spec}(A)$ is compact and sober, as the other conditions require more algebraic discussion (the spectra of quotient rings, finite ideals, etc.) which deviates from the rest of the exposition.

1. We use the corollary to soberification to show $\mathrm{Spec}(A)$ is sober: if $c : \mathfrak{p} \mapsto \overline{\{\mathfrak{p}\}}$ is bijective, then $\mathrm{Spec}(A)$ is sober. Now, we have from earlier that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. If $V(\mathfrak{p}) = V(\mathfrak{q})$, then $\mathfrak{p} \in V(\mathfrak{p})$ and $\mathfrak{q} \in V(\mathfrak{q}) \implies \mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{q} \subset \mathfrak{p}$, so $\mathfrak{p} = \mathfrak{q}$, so c is certainly injective. Then, since the irreducible subsets of $\mathrm{Spec}(A)$ are exactly the sets $V(\mathfrak{p})$, c must be surjective as well, so we win.

2. Note that since we have a basis for $\mathrm{Spec}(A)$, we reduce compactness to finding a finite subcover of
$$\mathrm{Spec}(A) = \bigcup_{i \in I} D(a_i) \text{ for } a_i \in A.$$
But note that taking complements yields that
$$\emptyset = \bigcap_{i \in I} V(a_i)$$
so there is so prime ideal that contains all of the a_i. But this directly implies that the ideal $\mathfrak{a}$ generated by the a_i cannot be proper (since it is not contained by a maximal ideal) so it must be all of A; then, we have that a finite combination of a_i must sum to 1, since the elements of $\mathfrak{a}$ are finite linear combinations of a_i. Reindexing, let $\sum_{i=1}^{n} c_i a_i = 1$.
Now, consider $\bigcap_{i=1}^{n} V(a_i)$; this is the set of all prime ideals that contain a_i. If $\mathfrak{p} \in \bigcap_{i=1}^{n} V(a_i)$, we would have that $a_i \in \mathfrak{p} \implies \sum_{i=1}^{n} c_i a_i = 1 \in \mathfrak{p} \implies \mathfrak{p} = A$, which is

a contradiction since prime ideals are proper. Thus, $\bigcap_{i=1}^n V(a_i)$ is empty, so once we take complements, we arrive at

$$\mathrm{Spec}(A) = \bigcup_{i=1}^{n} D(a_i)$$

□

This fully shows one direction of our desired equivalence:the spectrum of a ring is spectral. We leave it it Hochster to take care of the other direction:

Theorem 3 (Hochster 1969)**.** *Let* **Spec** *be the subcategory of* **Top** *consisting of spectral spaces and maps. Then, there is a functor* F : **Spec** → **CRing** *such that the composition of* Spec *and* F *is isomorphic to the inclusion functor* **Spec** → **Top***.*

The proof of this theorem is long and technical, but the rough outline is that Hochster starts out with some spectral space X and a ring A where X is dense in Spec(A), by constructing an algebra over some field k with such a spectrum. From there, he is able to massage A into some new ring which in fact does have X as its spectrum. Of course, the work done here is deeply technical and algebraically complicated, by the result is simple and beautiful, giving a conclusion to the relationship between the Zariski topologies and sober topological spaces. Not only can we describe a large subset of sober topological spaces by talking about commutative rings, we can actually even capture the continuous mappings between these spaces by considering ring homomorphisms, since the categories are actually isomorphic!

Acknowledgments

This document is written as a final project for Professor Mike Miller Eismeier's Fall 2021 undergraduate topology course at Columbia University. The author would like to thank Prof. Miller for not only his input throughout the project but also the excellent teaching throughout the semester, which has been indispensable to the understanding of the material in this exposition.

References

1. The Stacks Project: `https://stacks.math.columbia.edu/tag/004C`
2. R. Hartshorne, Algebraic Geometry
3. M. Hochster, *Prime ideal structure in commutative rings*