Metrization Theorems

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In the first few weeks of an intro topology course we are often taught to think of topology as "geometry without distances". We trade in our notion of distance for a more general notion of open sets, and we learn that this new topological framework, though weaker than the geometric one to which we are accustomed, nevertheless yields a lot of interesting information about a lot of spaces.

This approach, with its central step of constructing the metric topology and then ignoring the metric, carries the implication that the transaction is irreversible, that great topological power cannot come without deep geometric sacrifice. On some level, a topology seems to contain "more information" than a metric: given a space X, a metric is a single continuous map $X \times X \to [0, \infty)$; a topology encodes *all* the continuous maps to and from X. Surely it would be foolish optimism to suppose that we could convert topologies into metrics!

Indeed, there are many topological spaces which are not *metrizable*—that is, which cannot be constructed from metric spaces: for example, any set containing more than one element, with the indiscrete topology. But what is surprising—to me, at least—is how little is required in order to make a space metrizable. In this paper, we will discuss the Urysohn metrization theorem, which tells us that *every regular Hausdorff second-countable space is metrizable*, and then upgrade to the Nagata-Smirnov metrization theorem, which gives slightly more general conditions which turn out to be not only sufficient but necessary.

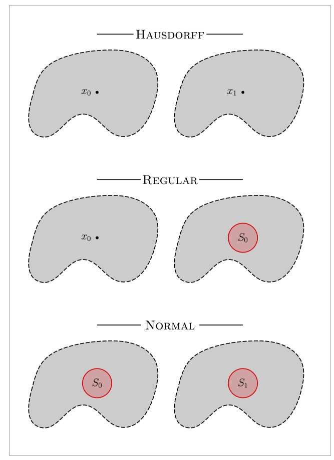
The Urysohn lemma

Our central challenge in proving metrizability will be to convert information about open sets in a space X into information about functions $X \to \mathbb{R}$. To do this we will use a theorem called the Urysohn lemma, an important and delightful result which Munkres' *Topology* describes (on page 207!) as "the first deep theorem of the book". First, however, we will remind ourselves of a few definitions which will express the notions of "separability" upon which we will rely.

Definition. A topological space X is called **Hausdorff** or T_2 if *points in X are separated by neighborhoods*: that is, if given distinct points a and b we can construct open sets $U_a \ni a$, $U_b \ni b$ such that $U_a \cap U_b = \varnothing$.

Definition. A topological space X is called **regular Hausdorff** or T_3 if X is Hausdorff and *points and closed sets in X are separated by neighborhoods*: that is, if given a point a and a closed set B we can construct open sets $U_a \ni a$, $U_B \supset B$ such that $U_a \cap U_B = \varnothing$.

Definition. A topological space X is called **normal Hausdorff** or T_4 if X is Hausdorff and *closed sets in X are separated by neighborhoods*: that is, if given disjoint closed subsets A and B we can construct open sets $U_A \supset A$, $U_B \supset B$ such that $U_A \cap U_B = \varnothing$.



Sets which are merely **regular** and **normal** obey the separation condition but need not be Hausdorff; for example, the two-point space with the indiscrete topology is regular and normal but not Hausdorff. It's straightforward to check that for regular or normal sets, Hausdorffness is equivalent to the condition that one-point sets are closed. Some authors, including Munkres, bundle Hausdorffness into their definitions of regularity and normality; we will not do this.

The proof of the Urysohn lemma will rely on a slightly different but equivalent definition of normality:

Definition. A topological space X is **normal** if, given a closed subset A_0 and an open subset U_2 which satisfy $A_0 \subset U_2$, we can find an "intermediate" open subset U_1 such that

$$A_0 \subset U_1 \subset \overline{U_1} \subset U_2.$$

To see that the definitions are equivalent, assume a space is normal under the second definition; given disjoint closed sets A_0 and B put $U_2 = B^C$. Then U_1 and $\overline{U_1}^C$ are neighborhoods of A_0 and B respectively.

Conversely, assume a space is normal under the first definition; given a closed subset A_0 and an open subset U_2 satisfying $A_0 \subset U_2$, we can put $B = U_2^C$ and let $U_1 \supset A_0$ and $U_B \supset B$ be neighborhoods containing them. Then $U_2^C = B \subset U_B \subset (U_1^C)^\circ = \overline{U_1}^C$, so $\overline{U_1} \subset U_2$.

Having finished that appetizer, we're ready for the main course.

Theorem (Urysohn lemma)**.** *Let X be a normal space, and let A and B be disjoint closed subsets of X. Then there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

Proof. We will first construct an infinite family of "nested" open sets U_p of X, indexed by the binary fractions; given these sets, we will have a natural way of defining a continuous function f.

Let P be a countable dense subset of $[0, 1]$ containing the endpoints: for example, we might use the rational numbers or the binary fractions. We want to define, for each $p \in P$, an open set $U_p \subset X$, in such a way that

$$p < q \Rightarrow \overline{U_p} \subset U_q. \tag{1}$$

P is countable, so we can do this inductively. Let P be indexed by a positive integer i in such a way that the first two elements are $p_1 = 1$, $p_2 = 0$. If we define $U_1 = B^C$, then we can use normality to choose an open set U_0 such that $A \subset U_0 \subset \overline{U_0} \subset U_1$.

We can then run inductively through P, doing this at each step. Suppose that we have already defined U_{p_i} for each $i < n$, in a way satisfying condition (1), and that we now wish to define U_{p_n}. Since we have only defined finitely many U_{p_i} so far, p_n is "moving in" between well-defined "neighbors" p and q: explicitly, there is some $p = \max\{p_i | p_i < p_n\}$ and some $q = \min\{p_j | p_j > p_n\}$, and $p < p_n < q$. Using normality, we can construct a set U_{p_i} such that

$$\overline{U_p} \subset U_{p_n} \subset \overline{U_{p_n}} \subset U_q.$$

By the transitivity of $<$ and $\subset$, the inclusion of U_{p_n} preserves condition (1).

We now extend our definition of U_p to all real numbers $p \in \mathbb{R}$. First, put

$$U_p = \varnothing \quad \text{if} \quad p < 0$$
$$U_p = X \quad \text{if} \quad p > 1.$$

Then it remains true that

$$p < q \Rightarrow \overline{U_p} \subset U_q.$$

Pick any point $x \in X$, and define

$$\mathcal{P}(x) = \{p | x \in U_p\}.$$

Note that if $p \in \mathcal{P}(x)$ and $q \geq p$, with $q \in P \cap (1, \infty)$, then $q \in \mathcal{P}(x)$. Now $\mathcal{P}(x)$ contains no number $p < 0$, since $U_p = \varnothing \not\ni x$ for $p < 0$, so it is bounded below by 0. On the other had, it contains every number $p > 1$, since $U_p = X \ni x$ for $p > 1$, so its greatest lower bound is at most 1. Define

$$f(x) = \inf \mathcal{P}(x) = \inf\{p | x \in U_p\} \subset [0, 1].$$

It is clear that $f(A) = \{0\}$ and $f(B) = \{1\}$; we claim that f is continuous.

To see why, consider an open interval $(p, q) \subset [0, 1]$, with $p, q \in P$: we wish to show that $f^{-1}((p, q))$ is open. We claim that $f^{-1}((p, q)) = U_q \backslash U_p$. Consider the following three cases:

- Suppose $x \in U_q \backslash U_p$. We know that $x \in U_q$ but $x \notin U_p$, so $q \in \mathcal{P}(x)$ but $p \notin \mathcal{P}(x)$. Hence $p < \inf \mathcal{P}(x) < q$: that is, $f(x) \in (p, q)$.
- Suppose $x \notin U_q$. We know that $x \notin U_p$ for $p < q$, so $\inf \mathcal{P}(x) \geq q$: that is, $f(x) \in [q, \infty)$.
- Suppose $x \in U_p$. Then $\inf \mathcal{P}(x) \leq p$: that is, $f(x) \in (-\infty, p]$.

This proves that $f^{-1}((p, q)) = U_q \backslash U_p$ for $p, q \in P$. Because P is dense in $[0, 1]$, these open intervals (p, q) form a basis on $[0, 1]$, so the preimage of any open set in $[0, 1]$ is open and f is continuous. $\square$

Incidentally, the converse is also true (just take $U_A = f^{-1}([0, \frac{1}{2}))$ and $U_B = f^{-1}((\frac{1}{2}, 1])$, for example), so the Urysohn lemma shows that a set is normal *iff* closed sets are separated by continuous functions.

The Urysohn metrization theorem

We're ready for our main theorem! Using our newfound power to construct functions separating open sets, we will create an embedding of our space into $\mathbb{R}^\infty$. In particular, in a normal Hausdorff space, given any point x and neighborhood $U \ni x$, we can use the Urysohn lemma to construct a function which is positive at x but vanishes outside U. We might try to simply do this for every point and neighborhood, assigning each function to a different coordinate, but we have only countably many coordinates to work with and may well have uncountably many open sets. We'll therefore also require that X have a countable basis, and declare that...

Theorem (Urysohn metrization theorem)**.** *Every normal Hausdorff second-countable space X is metrizable.*

Proo—hold on! We can do better than this! As it turns out, *every regular second-countable space is normal* (see the end of the paper for a proof of this). We can therefore strengthen the theorem:

Theorem (Urysohn metrization theorem, for real this time)**.** *Every regular Hausdorff second-countable space X is metrizable.*

Proof. We first prove the following:

Proposition. *There exists a countable collection of continuous functions $f_n : X \to [0,1]$ such that, for any point $x \in X$ and neighborhood $U \ni x$, there is some n such that $f_n(x) > 0$ but $f_n(U^C) = \{0\}$.*

Proof. If we were satisfied with just any collection—rather than specifically a countable one—we could just use the Urysohn lemma directly on each pair (x, U). But, as mentioned, an uncountable collection would be too big to make the next step work. Instead, for each such pair, use normality to pick basis sets B, B' such that

$$x \in B \subset \overline{B} \subset B' \subset U.$$

(If you're not convinced that this can be done: pick B' first; then pick a set U satisfying $x \in U \subset \overline{U} \subset B'$, without worrying about making it a basis set. Finally choose a basis set $B \subset U$.) Now use the Urysohn lemma on B'^C and B to find a function $f_{x,U}$ such that $f_{x,U}(B'^C) = \{0\}$ and $f_{x,U}(B) = \{1\}$. In particular, $f_{x,U}(x) = 1$ and $f_{x,U}$ vanishes outside U. There are only countably many pairs (B, B'), so we can rename these f_n, indexed by the natural numbers. □

We return to the proof of the main theorem. We now have a collection of countably many functions $f_n : X \to \mathbb{R}$; we will use them as components, defining a map $F : X \to \mathbb{R}^\infty$ by

$$F(x) = (f_1(x), f_2(x), \ldots).$$

We wish to show that F is an embedding.

Continuity is easy: F is continuous because each of the components f_n is continuous and $\mathbb{R}^\infty$ has the product topology. Injectivity follows from the way we defined the f_n: given $x \neq y$, choose a neighborhood U of x such that $y \notin U$; then there is some n for which $f_n = f_{x,U}$, so $f_n(x) = 1 \neq 0 = f_n(y)$.

The difficult step is to prove that F is an open map onto its image $Z = F(X)$, i.e., that an open set U in X has an image $F(U)$ which is open in Z. Given an open set U, we will go about this as follows: we will pick an arbitrary point $z_0 \in F(U)$ and then show that we can fit it in an open set W of Z such that

$$z_0 \in W \subset F(U).$$

Pick out the point $x_0 \in U$ such that $F(x_0) = z_0$, and choose the n for which $f_n = f_{x_0,U}$, i.e., $f_n(x_0) = 1$ and $f_n(U^C) = \{0\}$. Then

$$y \notin U \Rightarrow f_n(y) = 0$$

so

$$f_n^{-1}((0, +\infty)) \subset U. \tag{2}$$

Define $W = \pi_n^{-1}((0, +\infty)) \cap Z$. This is open in Z because $\pi^{-1}((0, +\infty))$ is open in $\mathbb{R}^\infty$ (since it's the inverse image of an open set). Now $W \subset F(U)$ by equation (2), and $z_0 \in W$ because $f_n(x_0) = 1 > 0$. Hence W is an open neighborhood of z_0 contained in $F(U)$. Since z_0 was chosen arbitrarily, $F(U)$ must be open.

We've now shown that F is a continuous bijection which is open on its image: hence F is an embedding of X in $\mathbb{R}^\infty$. Then F is homeomorphic to its image $Z = F(U)$, which is a subspace of a metric space and therefore itself a metric space. $\square$

The Nagata-Smirnov metrization theorem

Our proof of the Nagata-Smirnov metrization theorem will be an "upgrade" of our proof of the Urysohn metrization theorem, in which we will relax our conditions somewhat and modify the proof to make it work in the more general setting.

Before stating the theorem we will need a few definitions.

Definition. A collection $\mathcal{B}$ of subsets of a topological space X is called **locally finite** if every point of X has a neighborhood that intersects only finitely many of the sets in $\mathcal{B}$.

Definition. A collection $\mathcal{B}$ of subsets of a topological space X is called **countably locally finite** if it is the union of countably many subcollections $\mathcal{B}_n$, each of which is locally finite.

Definition. A set A is referred to as a "G_δ set" if A is a countable intersection of open sets.

It turns out to be true that regular spaces with countably locally finite bases are normal, and that any closed set in such a space is G_δ. The proof is similar to the proof that regular second-countable spaces are normal, given at the end of the paper; we will not go over it in detail.

The proof of the Urysohn metrization theorem relied on constructing functions which vanished outside open sets and took positive values on smaller regions inside those sets. For Nagata-Smirnov, we will want to make this more precise: we would like to construct functions which vanish outside an arbitrary open set and are positive *everywhere* inside. The notion of a G_δ set gives us a useful lemma which will provide us with a way of doing this.

Lemma. *Let A be a closed G_δ set in a normal space X. Then there is a continuous function $f : X \to [0, 1]$ such that*

$$f(x) = 0 \text{ for } x \in A, \qquad f(x) > 0 \text{ for } x \notin A.$$

Proof. By G_δ-ness, we can write A as the intersection of countably many open sets U_n. For each n use the Urysohn lemma to choose $f_n : X \to [0, 1]$ such that $f_n(A) = \{0\}$ and $f_{n(U_n^C} = \{1\}$. Then define

$$f(x) = \sum_n 2^{-n} f_n(x).$$

This goes to $[0, 1]$ as it should, and it's positive on A^C while vanishing on A. It's also continuous, as each term is continuous and the series converges uniformly. □

We're finally ready to state the theorem.

Theorem. *(Nagata-Smirnov metrization theorem) A topological space X is metrizable if and only if X is regular and has a countably locally finite basis.*

Proof. We'll do the $\Leftarrow$ direction first. Pick a countably locally finite basis $\mathcal{B}$ for our space X. Decompose $\mathcal{B}$ as a countable union of locally finite families $\mathcal{B}_n$:

$$\mathcal{B} = \bigcup_n \mathcal{B}_n.$$

For each n and each set $B \in \mathcal{B}_n$, use the lemma above to choose a continuous function $f_{n,B} : X \to [0, \frac{1}{n}]$ such that

$$f_{n,B}(x) > 0 \text{ for } x \in B, \qquad f_{n,B}(x) = 0 \text{ for } x \notin B.$$

(To be precise: apply the lemma to B^C, which is closed, and then divide the result by n.)

We define a set $J = \{(n, B) | n \in \mathbb{Z}_+, B \in \mathcal{B}_n\}$. Note that J need not be countable! Now define a function $F : X \to [0, 1]^J$ by its components:

$$\pi_{n,B} \circ F = f_{n,B} \text{ for each } (n, b) \in J.$$

In the proof of the Urysohn metrization theorem, we proved that a very similar function was an embedding into $[0,1]^\infty$, when the codomain was taken as having the product topology. That proof did not rely on the fact that the codomain was only countable-dimensional, so we can take the same approach to show that F is an embedding relative to the product topology on $[0,1]^J$.

Unfortunately, the product topology with uncountably many dimensions does not give us a metric in a straightforward way (for instance, the Euclidean metric or a variant thereof would require adding one term per dimension, which doesn't make sense with uncountably many dimensions). Instead, we will prove that F is an embedding into $[0,1]^J$ with the topology generated by the *uniform metric*, in which $d_{\text{unif}}(p,q) = \sup_{j\in J}(|p_j = q_j|)$. The uniform metric topology is finer than the product topology, i.e., open sets in the product topology are always open in the uniform metric topology, [1] so the fact that F is an open map onto its image in the product topology implies that this also holds in the uniform metric topology. What remains is to prove that F is continuous.

We will do this in the "analysis way": we'll take any point $x_0 \in X$ and, given any $\varepsilon > 0$, find a neighborhood W of x_0 such that

$$x \in W \Rightarrow d_{\text{unif}}(x, x_0) < \varepsilon.$$

Here is where we use the fact that $f_{n,B}(x) \in [0, \frac{1}{n}]$. Choose N such that $\frac{1}{N} < \frac{\varepsilon}{2}$. For $n \geq N$, we automatically have

$$|f_{n,B}(x) - f_{n,B}(x_0)| \leq \frac{1}{n} < \frac{\varepsilon}{2}$$

Now we have only finitely many n to deal with. For each $n < N$ do the following: first, using the fact that $\mathcal{B}_n$ is locally finite, choose a neighborhood U_n of x_0 which intersects only finitely many of the sets in B_n. Call these sets B_i. Each f_{n,B_i} is continuous, so for each i choose a neighborhood $V_{n,i}$ of x_0 such that

$$|f_{n,B_i}(x) - f_{n,B_i}(x_0)| < \frac{\varepsilon}{2}$$

for all $x \in V_{n,i}$. Because there are only finitely many $V_{n,i}$, we can take their intersection to get a new neighborhood $V_n = \cap_i V_{n,i}$ such that for $x \in V_n$

$$|f_{n,B}(x) - f_{n,B}(x_0)| < \frac{\varepsilon}{2} \quad \text{for } all\ B \in \mathcal{B}_n.$$

(This works for the B_i because of how we defined V_n, and it works for the other $B \in \mathcal{B}_n$ because x is zero, and hence constant, on those sets.)

Having produced a neighborhood V_n of x_0 in this way for every $n < N$, we once again take a finite intersection, putting $W = \cap_{n<N} V_n$. Now, for each $x \in W$, we have

$$|f_{n,B}(x) - f_{n,B}(x_0)| < \frac{\varepsilon}{2}$$

[1] Why? Take a basis element U of the product topology. U is the Cartesian product of finitely many balls, $U_i \in \mathbb{R}$, and infinitely many copies of the whole of $\mathbb{R}$. Pick any $p \in U$. Choose $\varepsilon > 0$ small enough that $y_i \in U_i$ whenever $|x_i - y_i| \leq \varepsilon$, for each i. Now $B_{\varepsilon,\text{unif}} \subset U$. (Note that we can do this because there are only finitely many U_i; this would not work for the box topology, which is in fact finer than the uniform metric topology.)

for *all* $(n, B) \in J$. Therefore

$$\sup\left(|f_{n,B}(x) - f_{n,B}(x_0)|\right) \leq \frac{\varepsilon}{2} < \varepsilon$$

for all $x \in W$. This satisfies the definition of continuity, completing the proof that F is an embedding. Since the codomain is (by definition) a metric space, X is metrizable!

We will now prove the converse, using our newfound metric-space superpowers to efficiently undo all that hard work. We first need a lemma:

Lemma. *Let X be a metrizable space. If $\mathcal{A}$ is an open cover of X, there exists a refinement of $\mathcal{A}$ which is countably locally finite.*

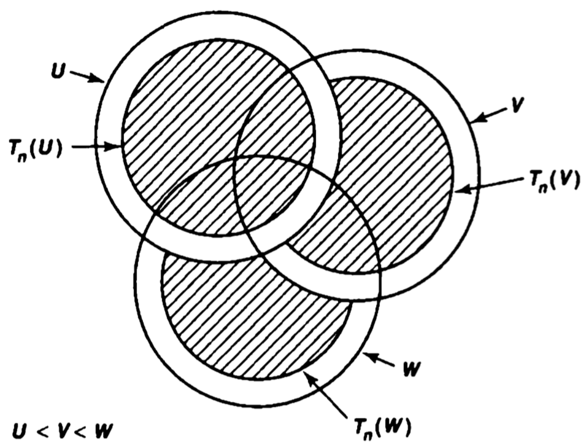
Proof. Pick any integer n. Denote the sets in $\mathcal{A}$ by $U, V, W, \ldots$, and "shrink" them all by $\frac{1}{n}$:

$$S_n(U) = \{x \in U | B_{1/n}(x) \subset U\}$$

Using the well-ordering theorem, choose an (arbitrary) ordering of the sets in $\mathcal{A}$. Construct a new family of sets by subtracting from each $S_n(U)$ all the sets $V \in A$ which precede U in the ordering:

$$T_n(U) = S_n(U) - \bigcup_{V \in \mathcal{A}, V \prec U} V.$$

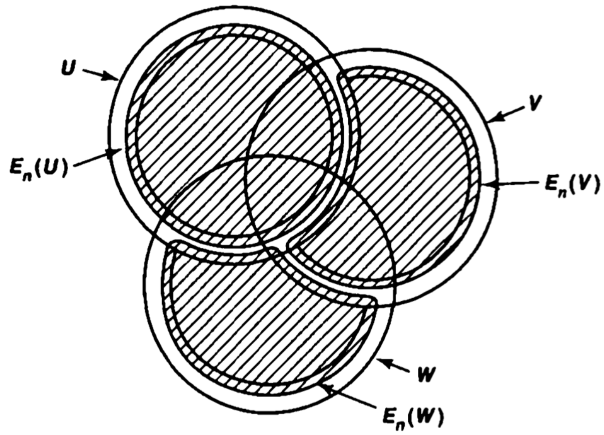
Note that points in different sets $T_n(U), T_n(V)$ are separated by a distance of at least $\frac{1}{n}$. We now enlarge



the sets $T_n(U)$ in order to make them open. Define

$$E_n(U) = \bigcup_{x \in T_n(U)} B_{\frac{1}{3n}}(x).$$

Now put $\mathcal{E}_n = \{E_n(U) | U \in \mathcal{A}\}$. This is a locally finite family of sets: by the triangle inequality, the



neighborhood $B_{\frac{1}{6n}}(x)$ of any point x intersects with at most one set $E_n(U)$. We can construct such a set $\mathcal{E}_n$ for every positive integer n and then take their union,

$$\mathcal{E} = \bigcup_{n>0} \mathcal{E}_n.$$

$\mathcal{E}$ covers X, for the following reason. Pick a point $x \in X$ and let $U \in \mathcal{A}$ be the first set in the ordering which contains x. Now, because U is open, we can choose a distance ε such that $B_\varepsilon(x) \subset U$. Choose n sufficiently large that $\frac{1}{n} < \varepsilon$: then we know that $x \in S_n(U)$. We also know that $x \notin V$ for any $V \prec U$, so $x \in T_n(U) \subset E_n(U)$. We can do this with any $x \in X$, so $\mathcal{E}$ is an open cover. □

We now use this lemma to complete the proof. For each integer $m > 0$ take $\mathcal{A}_m = \{B_{1/m}(x) | x \in X\}$. Use the lemma to refine $\mathcal{A}_m$ to a countably locally finite cover $\mathcal{B}_m$. We claim that $\mathcal{B} = \bigcap_m \mathcal{B}_m$ is a basis.

Given any $\varepsilon > 0$ and $x \in X$, we want to find an element of $\mathcal{B}$ which is contained in $B_\varepsilon(x)$. Choose m large enough that $\frac{1}{m} < \frac{\varepsilon}{2}$, and choose any set $B \in B_m$ such that $x \in B$. By the triangle inequality, points in B are at most a distance ε apart, so $B \subset B_\varepsilon(x)$.

Therefore $\mathcal{B}$ is a countably locally finite basis for X.

□

Appendix: regular second-countable spaces are normal

Proposition. *Every regular second-countable space is normal.*

Proof. Let $\mathcal{B}$ be a countable basis for X, and let A, B be closed sets in X. For each point $a \in A$, by regularity, we can choose an open neighborhood $U_a \ni a$ which is a basis element ($U_a \in \mathcal{B}$) such that $\overline{U_a} \cap B = \varnothing$.

Then we can take the union

$$A \subset \bigcup_{a \in A} U_a.$$

Similarly, for each $b \in B$, we can choose an open neighborhood $V_b \ni b$ which is a basis element, such that $\overline{V_b} \cap A = \varnothing$, and take the union

$$B \subset \bigcup_{b \in B} V_b.$$

By countability, both of these are countable unions, so we can re-index the sets:

$$A \subset \bigcup_{a \in A} U_a = \bigcup_{n \in N} U_n \quad \text{and} \quad B \subset \bigcup_{b \in B} V_b = \bigcup_{n \in N} V_n$$

These unions *almost* work as open neighborhoods separating A and B, but not quite: they might intersect. But we can change our neighborhoods slightly to fix this. For each $n \in \mathbb{N}$, let

$$U_n' = U_n \setminus \bigcup_{k=1}^{n} \overline{V_k} \quad \text{and} \quad V_n' = V_n \setminus \bigcup_{k=1}^{n} \overline{U_k}$$

The unions in the above are closed, because they are *finite* unions of closed sets; hence U_n' and V_n' are open. (This ability to relate each U_i' to only finitely V_i and vice versa is why we need to have rewritten the unions as finite unions, and ultimately why we need second-countability.) Now we can define

$$U = \bigcup_{n \in \mathbb{N}} U_n' \quad \text{and} \quad V = \bigcup_{n \in \mathbb{N}} V_n',$$

open sets which, we claim, are disjoint. To see why, pick a point $x \in U$: then $x \in U_k'$ for some k. By the way we defined the sets U_n', $x \notin V_i'$ for all $i \leq k$. On the other hand, for $i > k$, we defined the V_i' to satisfy $V_i' \cap \overline{U_k} = \varnothing$, so $x \notin V_i'$. Therefore $x \notin V$, so $U \cap V = \varnothing$. □

Attributions

The graphic showing Hausdorff, regular, and normal spaces is CC BY–SA 4.0 by Wikimedia user Osmium2356. The two graphics in the proof of the Nagata-Smirnov metrization theorem are from Munkres' *Topology*.