

MATH4051 Final Project: the Haar Measure

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1 Preliminaries: Topological groups, measures

1.1 Introduction

In this project, we will explore the Haar measure, an important idea that establishes a notion of “volume” for many important topological groups. For our purposes, a measure is a function which assigns a non-negative number to subsets of a space, which intuitively represents the “size” or “volume” of the subset. A measure on a space allows one to integrate real-valued functions from that space with respect to that measure.

Topological groups are a natural construct arising from the basic notions of topology. Many of the most basic and important groups come with a

natural topological structure. The powerful construct of the Haar measure, introduced by Alfred Haar in 1933, shows that many of these groups come with an essentially unique (up to a scalar factor) measure. Specifically, if G is a topological group that is *locally compact* we may almost unambiguously define a theory of integration for continuous functions $f : G \rightarrow \mathbb{R}$. This notion of integration leads to a number of important applications, such as the representation theory and harmonic analysis of compact groups.

1.2 Topological Groups

In this section, we will review the definitions for the basic concepts related to the Haar measure: locally compact groups and measures. First, we introduce locally compact groups.

Definition 1. A **topological group** is a topological space (G, \mathcal{T}) equipped with a group structure (G, e, m, i) whose group operations are continuous with respect to the underlying topology. Specifically, we have continuous maps $m : G \times G \rightarrow G$, $i : G \rightarrow G$, and a distinguished element $e \in G$ satisfying the following properties for all $a, b \in G$:

- $m(a, m(b, c)) = m(m(a, b), c)$,
- $m(a, e) = m(e, a) = a$,
- $m(a, i(a)) = m(i(a), a) = e$.

As usual, we will write $a \cdot b$, ab , or $a + b$ to denote the group operation $m(a, b)$, and a^{-1} for the inverse $i(a)$. The above definition is quite natural, as we require the topology of a topological group to interact nicely with its group operations.

Examples:

- (a) The real line $(\mathbb{R}, +)$ with the standard topology is a topological group under addition. This follows from the fact that addition $(x, y) \mapsto x + y$ and negation $x \mapsto -x$ are both continuous with respect to the standard topology.
- (b) The circle $(S^1, \cdot) = \{z \in \mathbb{C} : |z| = 1\}$ with the standard subspace topology is a topological group under complex multiplication. Again,

this is a simple consequence of the fact that the maps $(z, w) \mapsto zw$ and $z \mapsto 1/z$ are continuous maps from S^1 to S^1 in the standard topology.

We now consider the topological property that allows for the construction of a Haar measure:

Definition 2. A topological space (X, \mathcal{T}) is **locally compact** if every point $x \in X$ is contained in some compact neighborhood. Explicitly, for every $x \in X$, there exists an open set U and a compact set K satisfying $x \in U \subseteq K$. A locally compact group is a topological group G whose underlying topological space is locally compact and Hausdorff.

The condition that G is Hausdorff is a technical requirement related to the type of measure that we wish to construct. However, since all T_1 topological groups are Hausdorff, this is not much of a restriction in practice.

Examples:

- (a) The group $(\mathbb{R}, +)$ with the standard topology is locally compact: for any $x \in \mathbb{R}$, we see that $[x - 1, x + 1]$ is a compact neighborhood of x .
- (b) Any compact group G , by which we mean a Hausdorff topological group whose underlying topological space is compact, is locally compact. Indeed, the compact set G is a compact neighborhood of any $a \in G$. It follows that (S^1, \cdot) is locally compact, since S^1 is a compact subset of \mathbb{C} .

1.3 Measures

Now, we turn our attention to the concept of a measure. As mentioned previously, a measure on a space X is a function that assigns a non-negative value to certain subsets of X . The collection of subsets on which a measure is defined must form a σ -algebra:

Definition. A collection of subsets $\mathcal{A} \subset \mathcal{P}(X)$ is called a **σ -algebra** on X if \mathcal{A} satisfies the following properties:

- $X \in \mathcal{A}$,
- If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$,

- The collection \mathcal{A} is closed under countable unions: for any sequence $\{A_i\}_{i=1}^{\infty}$, if each $A_i \in \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$,
- The collection \mathcal{A} is closed under countable intersections: for any sequence $\{A_i\}_{i=1}^{\infty}$, if each $A_i \in \mathcal{A}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$.

These axioms will define a system of *measurable* sets and define the kinds of operations we may perform with our measure. The first axiom tells us that the space as a whole should be measurable, a reasonable assumption. The second axiom tells us that if a set is measurable, its complement should be as well. The third and fourth axioms allow us to take not only finite, but also countable unions and intersections of measurable sets. The possibility of taking countable operations is important for allowing measures to work with limits.

Example: For any set X , it is clear that the collection $\mathcal{P}(X)$ is a σ -algebra.

Remark: It is easy to check that given a collection of σ -algebras $\{\mathcal{A}_i\}_{i \in I}$, the intersection $\bigcap_{i \in I} \mathcal{A}_i$ is once again a σ -algebra. As a result, for any collection of subsets $\mathcal{S} \subseteq \mathcal{P}(X)$, one can take the intersection of all σ -algebras \mathcal{A} such that $\mathcal{S} \subseteq \mathcal{A}$. Letting $\mathcal{A} = \mathcal{P}(X)$, at least one such σ -algebra exists, so by taking the intersection of all of these σ -algebras, we may define $\sigma(\mathcal{S})$ to be the smallest σ -algebra satisfying $\mathcal{S} \subseteq \mathcal{A}$.

Key Example: Given any topological space (X, \mathcal{T}) , we define the *Borel sets* to be $\mathcal{B}(X) = \sigma(\mathcal{T})$, the σ -algebra generated by the open sets in X . From the properties of a σ -algebra, it is easy to see that \mathcal{B} contains all open subsets of X , all closed subsets, and any set that can be iteratively defined using countably many unions and intersections of open and closed sets.

We note that the first two properties imply that a σ -algebra must also contain \emptyset . With the notion of a σ -algebra, we can now define a measure.

Definition 3. Let $\mathcal{A} \subset \mathcal{P}(X)$. A **measure** on (X, \mathcal{A}) is a function

$$\mu : \mathcal{A} \rightarrow [0, +\infty]$$

satisfying the following properties:

- $\mu(\emptyset) = 0$,

- μ is countably additive: given a countable collection of pairwise disjoint sets, $\{A_i\}_{i=1}^{\infty}$, $A_i \cap A_j = \emptyset$ whenever $i \neq j$, we have

$$\mu \left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i).$$

If X is a topological space (X, \mathcal{T}) and the Borel sets $\mathcal{B}(X)$ are the domain of μ , we say μ is a Borel measure.

Measures are an important notion with many applications, which are useful whenever we wish to give a collection of subsets a notion of size. Perhaps the most important application is that given a measure $\mu : \mathcal{A} \rightarrow \mathbb{R}$ for a σ -algebra \mathcal{A} on X , one may define a notion of integration for a suitable class of so-called \mathcal{A} -measurable functions. Here, a function $f : X \rightarrow \mathbb{R}$ is \mathcal{A} -measurable if for all $c \in \mathbb{R}$,

$$\{x \in X : f(x) < c\} \in \mathcal{A}.$$

For all such functions, there is a standard and simple way of defining the integral $\int f d\mu$ (which may not exist when the integral is unbounded).

Example 1. We now turn to a key example which brings together all of the ideas discussed so far. The *Lebesgue measure*, which we will denote by λ , on \mathbb{R} is a Borel measure that formalizes our intuitive notion of length. The Lebesgue measure assigns any interval its length,

$$\lambda([a, b]) = b - a.$$

The construction of the Lebesgue measure begins by defining the so-called outer measure λ^* on all subsets of \mathbb{R} by

$$\lambda^*(A) = \inf \left\{ \sum_{i=1}^{\infty} b_i - a_i : A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i) \right\}$$

One then says a subset $B \subset \mathbb{R}$ is Lebesgue measurable if for all subsets $A \subset \mathbb{R}$, the following condition holds:

$$\lambda^*(A) = \lambda^*(A \cap B) + \lambda^*(A \cap B^c)$$

and define the Lebesgue measure λ to be the restriction of λ^* to the Lebesgue measurable subsets. This condition excludes certain pathological sets that

will break some of the measure properties, but allows for all Borel sets (plus some) to be Lebesgue measurable. We refer to (Cohn 2013) for the full proof that λ is a measure.

A key property of λ is that it is invariant with respect to shifting by a constant. If $c \in \mathbb{R}$ and A is Lebesgue measurable, we have

$$\lambda(c + A) = \lambda\{c + x : x \in A\} = \lambda(A).$$

As a simple result of this fact, the functional mapping integrable functions to their integrals $f \mapsto \int f d\lambda$ is also shift-invariant:

$$\int f(x) d\lambda = \int f(x - c) d\lambda$$

for all $c \in \mathbb{R}$, f integrable. Recalling that \mathbb{R} is a locally compact group under addition, we see that the Lebesgue measure respects the group operation of \mathbb{R} in an important way. This motivates the following definitions:

Definition 4. A measure μ on a topological group G is **left-invariant** if for every measurable subset $A \subset G$ and $s \in G$, if we denote

$$sA = \{sx : x \in A\}$$

then we have $\mu(sA) = \mu(A)$.

If f is a function $f : G \rightarrow \mathbb{R}$, we may define a **left-translate** sf by $(sf)(x) = f(s^{-1}x)$. If, in the above definitions, we multiply by s or s^{-1} from the right, we define **right-invariant measures** and **right-translates**.

Before moving on to the construction of the Haar measure, we state one last result concerning the existence of well-behaved measures on locally compact spaces. First, a couple more preliminary definitions:

Definition. Let X be a topological space. A function $f : X \rightarrow \mathbb{R}$ has **compact support** if there exists a compact subset K such that

$$\text{supp}(f) = \{x \in X : f(x) \neq 0\} \subseteq K.$$

On a space X , we use $\mathcal{K}(X)$ to refer to the set of continuous functions on X with compact support.

Definition. A measure μ on a topological space X is **regular** if it satisfies the following conditions:

- $\mu(K) < \infty$ is finite for all compact sets $K \subset X$
- For every set $A \subseteq X$,

$$\mu(A) = \inf\{\mu(U) : U \text{ an open set such that } A \subseteq U\}$$

- For every open set $O \subseteq X$,

$$\mu(O) = \sup\{\mu(K) : K \text{ a compact set such that } K \subseteq O\}$$

A regular measure behaves according to our intuitive notions of what the “volume” of sets in a topological space should be. Under a regular measure, compact sets have finite measure, which accords with the intuition that compactness is a condition related to finiteness. And furthermore, the measure of an arbitrary set A is approximated by the open sets that contain A , and the measure of an open set U is approximated by the measure of the compact sets contained inside of U . It is easy to see that the Lebesgue measure is regular. With these notions defined, we have the following result:

Theorem 1. (*Riesz-Markov Representation Theorem*) *Let X be a locally compact Hausdorff space, and I a linear functional on $\mathcal{K}(X)$ satisfying $I(f) \geq 0$ for non-negative $f \geq 0$. Then there is a unique regular Borel measure μ on X such that*

$$I(f) = \int f d\mu$$

for all $f \in \mathcal{K}(X)$.

On a given topological space X , if we wish to define a reasonable notion of integration for functions $f : X \rightarrow \mathbb{R}$, continuous functions with compact support are probably the minimal set of functions that should be integrable. For any measure μ such that these functions are integrable, the map

$$f \mapsto \int f d\mu$$

would provide a linear functional on $\mathcal{K}(X)$. The above results says that if we already *have* such a functional, and know what the integral of functions in $\mathcal{K}(X)$ should be, there is a unique measure, which is well-behaved, that corresponds to this notion of integration.

We will not present a proof here, but the idea is that to define a regular Borel measure μ , we must determine what the value $\mu(O)$ should be for open sets O . But we must have

$$\mu(O) = \int \chi_O d\mu$$

where χ_O is the characteristic function of O . We may not get this value from the linear functional I , as χ_O need not be continuous or have compact support, but we may approximate χ_O by continuous functions with compact support. In the case of $X = \mathbb{R}$, for example, if $O = (a, b)$, we may approximate χ_O by functions taking on the value 1 on sets $[a + \epsilon, b - \epsilon]$, 0 on $(-\infty, a) \cup (b, \infty)$, and linearly interpolating in between so that f is continuous. For general X , if we define

$$\mu^*(O) = \sup\{I(f) : f \in \mathcal{K}(X), f \leq \chi_O\}$$

then μ^* gives an outer-measure on the Borel sets of X , and we may use this to construct a regular Borel measure on X .

So using this result, instead of directly constructing a measure on a locally compact topological group G , we may construct a linear functional on $\mathcal{K}(X)$. We now have all the tools we need to state the main result.

2 The Haar Measure

Theorem 2. *If G is a locally compact group, there exists a left-invariant nonzero Borel measure, μ on G , which we call the **left Haar measure**. This measure is unique up to multiplication by positive real constants: if ν is another left-invariant nonzero measure on G , $\nu = c\mu$ for some $c > 0$.*

We note that a similar statement holds for a right-invariant measure, called the *right Haar measure*. For abelian groups, the notions are identical, and we shall see that for compact groups, the two measures coincide once again. In general, the two measures may be different, though there is always a connection between the two.

This is a powerful and remarkable result. As we saw above, the Lebesgue measure in \mathbb{R}^d is left-invariant and thus a Haar measure. To respect the group structure of \mathbb{R}^d amounts to preserving measure under translation. Thus, the Lebesgue measure formalizes our natural intuition of how one should

naturally measure the volume of subsets of \mathbb{R}^d : if we simply shift a set over, its volume should not change. The Haar measure represents a vast generalization, stating that any locally compact group G comes with two related, effectively unique notions of volume corresponding to the left and right Haar measures. Indeed, if G is compact and we normalize $\mu(G) = 1$, there *is* a unique measure μ .

At first sight, the Haar measure is surprising and seems to exist as if by magic. There is an anecdote that the great mathematician John Von Neumann “tried to talk Haar out of the Haar measure” (Pedersen 2000). However, with some thought one may develop some intuition for why such a measure should exist. A topological group G is “homogeneous” in its topological structure; topologically the points of G are all the same, as there is a homeomorphism taking every point to every other point. But it is not at all obvious that this fact can be realized concretely in an invariant measure, and the power and generality of the above statement truly is remarkable.

2.1 Construction of Haar measure

We now develop a proof of the above statement, following an argument originally due to André Weil, as outlined in Leopoldo Nachbin’s *The Haar Integral*. We shall use the notation Nachbin II.8 to refer, for example, to chapter 2, section 8 of Nachbin’s text. Throughout this section, G is locally compact as in Theorem 2.

One key fact used in the argument is the Tychonoff theorem:

Theorem. (Tychonoff) *Let $\{X_i\}_{i \in I}$ be an arbitrary collection of topological spaces. Then the product space $\prod_{i \in I} X_i$ is compact in the product topology if and only if each X_i is compact.*

We take this result for granted, and refer to proofs in any comprehensive general topology textbook, such as (Munkres 2000). Next, we describe a technical lemma that gives an equivalent condition for compactness, following a standard compactness-style argument.

Lemma 3. *Nachbin II.8, Proposition 30: Let X be Hausdorff and $\{A_i\}$ an arbitrary collection of nonempty subsets with the property that for all A_i, A_j , there exists a set A_k satisfying $A_k \subseteq A_i \cap A_j$. Then X is compact if and only if for every such collection $\{A_i\}$, there exists a point $a \in X$ satisfying*

$$a \in \bigcap_{i \in I} \overline{A_i}$$

Proof. (\implies) If X is compact, let $U_i = (\overline{A_i})^c$. Suppose the $\{U_i\}$ formed an open cover of X . Then for some finite subcollection,

$$U_{i_1} \cup \dots \cup U_{i_n} = X.$$

But there exists $A_i \subseteq A_{i_1} \cap \dots \cap A_{i_n}$, so taking the closure then complements,

$$X = U_{i_1} \cup \dots \cup U_{i_n} \subseteq U_i$$

which implies $U_i = (A_i)^c = X$, a contradiction since we assumed each A_i is non-empty. So the collection $\{U_i\}$ is not an open cover, and there exists $a \notin \bigcup_{i \in I} U_i \implies a \in \bigcap_{i \in I} \overline{A_i}$.

(\impliedby) Suppose X has the property stated in the lemma. Let $\{U_i\}$ be an arbitrary open cover. Let $\{A_i\}$ be the collection of the complements of finite unions over the U_i :

$$A_i = (U_{i_1} \cup \dots \cup U_{i_n})^c$$

Each A_i is closed, so $A_i = \overline{A_i}$. By de Morgan's law, we can see that if A_i and A_j are sets of this form, then so is $A_i \cap A_j$. But since $\{U_i\}$ is an open cover, each point $x \in X$ is in some U_k , and is excluded from the corresponding $A_k = U_k^c$. So the intersection $\bigcap_{i \in I} A_i$ is empty, and to avoid violating the property, we must have some A_i is empty. Then

$$A_i = (V_{i_1} \cup \dots \cup V_{i_n})^c = \emptyset$$

yields the finite subcover $\{V_{i_1}, \dots, V_{i_n}\}$. \square

The next lemma proves a fact that is key to our definition of the measure, and again follows a standard compactness argument that leverages the compact support of f . We let $\mathcal{K}_+(G)$ denote the set of non-negative continuous functions on G with compact support.

Lemma 4. *Nachbin II.5, Lemma 1: Let G be locally compact, $f, g \in \mathcal{K}_+(G)$, and $g \neq 0$. Then there exist $s_1, \dots, s_n \in G$ and real coefficients $c_1, \dots, c_n > 0$ so that*

$$f \leq \sum_{i=1}^n c_i \cdot s_i g$$

Proof. Let K be a compact set containing the support of f , and $t \in G$ such that $g(t) > 0$. Then for any $s \in K$, we see the left-translate

$$(st^{-1}g)(s) = g(ts^{-1}s) = g(t) > 0,$$

so we may choose a large enough coefficient c_s so that $c_s \cdot (st^{-1}g)(s) > f(s)$ on some neighborhood V_s . Since K is compact, we may reduce the open cover $\{V_s\}_{s \in K}$ to a finite subcover $\{V_{s_1}, \dots, V_{s_n}\}$ with $c_i \cdot (s_i t^{-1}g) \geq f$ on V_{s_i} . Then we conclude

$$f \leq \sum_{i=1}^n c_i \cdot s_i t^{-1}g.$$

□

Now, if a left-invariant measure μ on G existed, then for every $g \neq 0$, by the previous lemma we would have

$$\int f d\mu \leq \int \sum_{i=1}^n c_i \cdot s_i g = \sum_{i=1}^n c_i \int g d\mu$$

Building on this idea, for every $f, g \in \mathcal{K}_+(X)$, $g \neq 0$, we *define* the number $(f : g)$ to be the infimum over all possible sums $\sum c_i$ satisfying the condition above:

$$(f : g) := \inf \left\{ \sum_{i=1}^n c_i : \exists s_1, \dots, s_n \text{ such that } f \leq \sum_{i=1}^n c_i \cdot s_i g \right\}$$

The number $(f : g)$ represents a rough estimate of the ratio between the integral of f and the integral of g . We derive some basic consequences of this definition:

Lemma 5. *For all $f, g \in \mathcal{K}_+(G)$ with $g \neq 0$:*

- (1) $(f : g) > 0$ if $f \neq 0$,
- (2) $(f : g) = (sf : g)$ for all $s \in G$,
- (3) $(f_1 + f_2 : g) \leq (f_1 : g) + (f_2 : g)$,
- (4) $(kf : g) = k(f : g)$ for any real $k > 0$,
- (5) $(f_1 : g) \leq (f_2 : g)$ if $f_1 \leq f_2$,
- (6) $(f : h) \leq (f : g)(g : h)$.

Proof. These properties are all straightforward to verify. For (1), since f and g have compact support, consider $a = \sup_{x \in G} f(x)$ and $b = \sup_{x \in G} g(x)$. Then for any coefficients c_i satisfying

$$f \leq \sum_{i=1}^n c_i \cdot s_i g \implies a \leq \sum_{i=1}^n c_i \cdot b$$

so $a/b \leq \sum c_i$. Taking the infimum over all possible c_i yields $a/b \leq (f : g)$.

For (2), we note that for any coefficients c_i ,

$$f \leq \sum c_i \cdot s_i g \implies sf \leq \sum c_i \cdot (ss_i)g,$$

so that $(sf : g) \leq \sum c_i$. Taking the infimum, $(sf : g) \leq (f : g)$. Substituting s^{-1} for s and sf for f yields $(f : g) \leq (sf : g)$, the equality follows.

For (3), if $f_1 \leq \sum c_i \cdot s_i g$ and $f_2 \leq \sum d_j \cdot t_j g$, adding these inequalities yields $(f_1 + f_2 : g) \leq \sum c_i + \sum d_j$. Taking the infimum yields the inequality.

For (4),

$$f \leq \sum c_i \cdot s_i g \iff f \leq k \sum c_i \cdot s_i g = \sum (kc_i) \cdot s_i g$$

yields $(kf : g) = k(f : g)$. For (5), if $f_2 \leq \sum c_i \cdot s_i g$, then $f_1 \leq \sum c_i \cdot s_i g$. It follows that $(f_1 : g) \leq (f_2 : g)$.

Finally for (6), suppose $f \leq \sum_{i=1}^n c_i \cdot s_i g$ and $g \leq \sum_{j=1}^m d_j \cdot t_j h$. Then $s_i g \leq \sum_{j=1}^m d_j \cdot s_i t_j h$, and we see

$$f \leq \sum_{i=1}^n c_i \cdot \sum_{j=1}^m d_j s_i t_j h = \sum_{i,j} c_i d_j \cdot s_i t_j h$$

So $(f : h) \leq \sum_{i,j} c_i d_j = \sum_i c_i \cdot \sum_j d_j \implies (f : h) \leq (f : g)(g : h)$. \square

Next, we choose a designated function $f^* \neq 0 \in \mathcal{K}_+(G)$ whose integral we wish to evaluate to 1. When G is compact, for example, a natural choice is the constant function $f^* = 1$. Then we *define*:

$$\mu_F(f) = \frac{(f : F)}{(f^* : F)}$$

for every $f, F \in \mathcal{K}_+(G)$ such that $F \neq 0$. Since $f^* \neq 0$, by (1) from the last lemma the denominator is non-zero. The following properties follow easily from the properties in the previous lemma:

Lemma 6. For all $f, g \in \mathcal{K}_+(G)$ with $g \neq 0$:

- (1) $\mu_F(f) > 0$ if $F \neq 0$,
- (2) $\mu_F(f) = \mu_F(sf)$,
- (3) $\mu_F(f_1 + f_2) \leq \mu_F(f_1) + \mu_F(f_2)$,
- (4) $\mu_F(kf) = k\mu_F(f)$ for all real $k > 0$,
- (5) $\mu_F(f_1) \leq \mu_F(f_2)$ if $f_1 \leq f_2$,
- (6) $1/(f^* : f) \leq \mu_F(f) \leq (f : f^*)$.

Proof. (1)-(5) are direct consequences of the corresponding properties from lemma 5. For (6), we have

$$(f^* : F) \leq (f^* : f)(f : F), \quad \text{and} \quad (f : F) \leq (f : f^*)(f^* : F),$$

from (6) in the previous lemma, which combine to give this inequality. \square

At this point, we are almost ready to construct the linear functional on $\mathcal{K}_+(G)$ which maps f to its integral $\mu(f)$. However, property (3) only gives us subadditivity, while the integral must be additive. But the next lemma shows that μ_F nearly becomes additive as the support of F becomes a small neighborhood of the identity.

Lemma 7. Given $f_1, f_2 \in \mathcal{K}_+(G)$ and $\epsilon > 0$, there exists a neighborhood V of the identity so that

$$\mu_F(f_1) + \mu_F(f_2) \leq \mu_F(f_1 + f_2) + \epsilon$$

for all $F \neq 0$ satisfying $\text{supp}(F) \subset V$.

Proof sketch. Suppose f_1 and f_2 are supported on some compact set K . Let f' be any function $f' \in \mathcal{K}_+(G)$ that equals 1 on K , then for some $\delta > 0$, define $f = f_1 + f_2 + \delta f'$. Then, we define h_1 and h_2 by

$$h_j(x) = \begin{cases} \frac{f_j(x)}{f(x)} & f(x) \neq 0 \\ 0 & f(x) = 0 \end{cases}$$

We note that $h_1 + h_2 \leq 1$. Also, each h_j is continuous on the open set $\{f(x) > 0\}$ and on K^c , since $f_j = 0$ on this latter set. These two open sets cover G , so h_j is continuous and has support contained by K .

Next, we claim that for any $\eta > 0$, there exists a neighborhood of the identity $e \in V$ so that $|h_j(x) - h_j(y)| < \eta$ whenever $x^{-1}y \in V$. This is a general result about continuous functions on G with compact support, and follows from a fairly typical compactness argument. Assuming this V exists, let $F \in \mathcal{K}_+(G)$ with $\text{supp}(F) \subset V$. Then if $f \leq \sum_i c_i \cdot s_i F$, we have

$$f(x)h_j(x) \leq \sum_i c_i \cdot F(xs_i^{-1}) \cdot h_j(x).$$

If $xs_i^{-1} \notin V$, then $F(xs_i^{-1}) = 0$, while if $xs_i^{-1} \in V$, then $h_j(x) \leq h_j(s_i) + \eta$. So we get

$$f_j(x) = f(x)h_j(x) \leq \sum_i c_i \cdot (s_i F)(x)(h_j(s_i) + \eta)$$

which implies

$$(f_j : F) \leq \sum_i c_i (h_j(s_i) + \eta)$$

Recalling $h_1 + h_2 \leq 1$, adding these inequalities for $j = 1, 2$ yields:

$$(f_1 : F) + (f_2 : F) \leq \sum_i c_i (1 + 2\eta)$$

Taking the infimum over all c_i satisfying $f \leq \sum_i c_i \cdot s_i F$, we have

$$(f_1 : F) + (f_2 : F) \leq (1 + 2\eta)(f : F)$$

Dividing by $(f^* : F)$ and applying properties (3) and (4) of the previous lemma, we have

$$\begin{aligned} \mu_F(f_1) + \mu_F(f_2) &\leq (1 + 2\eta)\mu_F(f) \\ &\leq (1 + 2\eta)(\mu_F(f_1 + f_2) + \delta\mu_F(f')) \end{aligned}$$

By property (6) of the previous lemma, $\mu_F(f')$ is bounded by $(f' : f^*)$. So expanding,

$$\mu_F(f_1) + \mu_F(f_2) \leq \mu_F(f_1 + f_2) + 2\eta\mu_F(f_1 + f_2) + \delta(1 + 2\eta)\mu_F(f')$$

we may take η and δ arbitrarily small to satisfy the lemma. \square

With the results above, we now turn to the existence part of Theorem 2.

Proof of Existence. Let $D = \{f \in \mathcal{K}_+(G) : f \neq 0\}$. Consider the space

$$J = \prod_{f \in D} [1/(f^* : f), (f^* : f)]$$

By Tychonoff's theorem, this set is compact. For every $F \in D$, by property (6) from lemma 6, there is a point $a_F = \{\mu_F(f)\}_{f \in D} \in J$ where we take $\mu_F(f)$ for the f -coordinate in J .

For any neighborhood V of the identity, with the definition of a_F as above, let

$$A_V = \{a_F : \text{supp}(F) \subset V\}.$$

Every A_V is non-empty, and whenever $V \subset V_1 \cap V_2$, we see $A_V \subset A_{V_1} \cap A_{V_2}$. So by lemma 3 above, there is a point $a \in \bigcap_V \overline{A_V}$, where we take the intersection over all neighborhoods V of e . This point $a \in J$ is described by a set of coordinates that assigns a positive real number a_f to each $f \in D$. Denote this coordinate $a_f = \mu(f)$. This number will *define* the integral of f .

The condition that a is in the intersection of all $\overline{A_V}$ gives us the following fact: for any $f_1, \dots, f_n, \epsilon > 0$, and neighborhood of the identity V , there exists $F \in D$ with $\text{supp}(F) \subset V$ and

$$|\mu(f_i) - \mu_F(f_i)| < \epsilon.$$

Using this fact and the properties from lemma 6, we get the following properties for all $f, f_1, f_2 \in D, s \in G, k > 0$:

- (i) $\mu(f) = \mu(sf)$
- (ii) $\mu(f_1 + f_2) = \mu(f_1) + \mu(f_2)$
- (iii) $\mu(kf) = k\mu(f)$
- (iv) $\mu(f) > 0$

We will give the argument for the second property: given f_1 and f_2 , by lemma 7 there exists a neighborhood V so that

$$|\mu_F(f_1 + f_2) - \mu_F(f_1) - \mu_F(f_2)| < \epsilon$$

whenever $F \in D$ and $\text{supp}(F) \subset V$. By the property above, there exists F supported on V satisfying $|\mu(g) - \mu_F(g)| < \epsilon$ for $g = f_1, f_2$, or $f_1 + f_2$. Then by the triangle inequality:

$$\begin{aligned} |\mu(f_1 + f_2) - \mu(f_1) - \mu(f_2)| &\leq |\mu(f_1 + f_2) - \mu_F(f_1 + f_2)| + |\mu_F(f_1) - \mu(f_1)| \\ &\quad + |\mu_F(f_1 + f_2) - \mu_F(f_1) - \mu_F(f_2)| \\ &\quad + |\mu_F(f_2) - \mu(f_2)| \\ &\leq 4\epsilon \end{aligned}$$

Taking $\epsilon \rightarrow 0$ gives $\mu(f_1 + f_2) = \mu(f_1) + \mu(f_2)$. Similar arguments prove the other properties.

Next, for the constant function $f = 0$, we define $\mu(f) = 0$, so that $f \mapsto \mu(f)$ is a well-defined, linear functional on $\mathcal{K}_+(G)$. Next, we show that we may extend μ to arbitrary (as opposed to non-negative) $f \in \mathcal{K}(G)$. We note that we may write $f = f_1 - f_2$ for some $f_1, f_2 \in \mathcal{K}_+(G)$ where

$$f_1(x) = \max(f(x), 0), \quad \text{and} \quad f_2(x) = \max(-f(x), 0).$$

It is simple to check that $f_1, f_2 \in \mathcal{K}_+(G)$. Then we define

$$\mu(f) = \mu(f_1) - \mu(f_2)$$

The linearity properties of μ on $\mathcal{K}_+(G)$ will carry over, so that $f \mapsto \mu(f)$ is a linear functional on $\mathcal{K}(G)$. This functional defines a left-invariant integral on $\mathcal{K}(G)$.

By Theorem 1 above, this functional corresponds to a unique regular, Borel measure μ . We recall that in the construction of this measure, for an open set O , we will find a sequence of functions $f_i \in \mathcal{K}(G)$ approximating the characteristic function $f_i \rightarrow \chi_O$, and letting $\mu(O) = \lim \mu(f_i)$. Since the functional μ is left-invariant, the resulting measure must be left-invariant as well. This completes the proof of existence. \square

This establishes that a left-invariant, regular, Borel measure on G exists whenever G is locally compact, already a nice result. Below, we sketch a proof for the uniqueness result of Theorem 2, relying on some more results from measure theory.

Proof of Uniqueness (sketch). Suppose μ and ν are left-invariant Borel measures on G . We use the following result about locally compact spaces: if $h : G \times G \rightarrow \mathbb{R}$ is continuous with compact support in $G \times G$, then

$$\iint h(x, y) d\mu(x) d\nu(y) = \iint h(x, y) d\nu(y) d\mu(x)$$

The above notation simply means iterated integration. For a proof that these integrals both exist and are equal, we once again refer to (Cohn 2013). Using the identity above and the left-invariance of μ , we have

$$\begin{aligned} \iint h(x, y) d\nu(y) d\mu(x) &= \iint h(y^{-1}x, y) d\mu(x) d\nu(y) \\ &= \iint h(y^{-1}x, y) d\nu(y) d\mu(x) \end{aligned}$$

Then substituting $y = xy$ and using left-invariance again:

$$= \iint h(y^{-1}, xy) d\nu(y) d\mu(x)$$

Now, let $f, g \in \mathcal{K}(G)$ and $g \neq 0$. Consider the function

$$h(x, y) = \frac{f(x)g(yx)}{\int g(tx) d\nu(t)}$$

Then h is well-defined since $\int g(tx) d\nu(t) = \int g(t) d\nu(t) > 0$ for all x , and we may easily check that h has compact support. Then we calculate:

$$\iint h(x, y) d\nu(y) d\mu(x) = \iint \frac{f(x)g(yx)}{\int g(tx) d\nu(t)} d\nu(y) d\mu(x) = \int f(x) d\mu(x)$$

Now, we use the identity from above to calculate the integral a second way:

$$\begin{aligned} \iint h(x, y) d\nu(y) d\mu(x) &= \iint h(y^{-1}, xy) d\nu(y) d\mu(x) \\ &= \iint \frac{f(y^{-1})g(x)}{\int g(ty^{-1}) d\nu(t)} d\nu(y) d\mu(x) \\ &= \left(\int g(x) d\mu(x) \right) \left(\int \frac{f(y^{-1})}{\int g(ty^{-1}) d\nu(t)} d\nu(y) \right) \end{aligned}$$

Equating these two calculations and dividing by $\int g d\mu$, we have

$$\frac{\int f(x) d\mu(x)}{\int g(x) d\mu(x)} = \int \frac{f(y^{-1})}{\int g(ty^{-1}) d\nu(t)} d\nu(y)$$

Since the RHS does not depend on μ , if we follow the same argument using ν instead of μ , we find

$$\begin{aligned} \frac{\int f d\mu}{\int g d\mu} &= \frac{\int f d\nu}{\int g d\nu} \\ \implies \int f d\mu &= \int f d\nu \left(\frac{\int g d\mu}{\int g d\nu} \right) \end{aligned}$$

so that the functional $f \mapsto \int f d\mu$ is simply a scalar multiple of the functional $f \mapsto \int f d\nu$. We may conclude (again by following through the construction of the measures corresponding to these functionals in Theorem 1) that $\mu = c\nu$ for some real scalar $c > 0$. \square

This completes the proof of Theorem 2. We note that the same argument with a few changes works to construct the right Haar measure on a locally compact group, which again is unique up to a scalar factor.

2.2 Basic properties

In this section we quickly present some basic properties of the Haar measure.

Proposition 8. *For a locally compact group G , let μ be a left-invariant measure. Then $\mu(G) < \infty$ is finite if and only if G is compact.*

Proof. (\Leftarrow) When G is compact, the constant function $f = 1$ is continuous with compact support, and hence integrable. So $\mu(G) = \int 1 d\mu < \infty$.

(\Rightarrow) Suppose G is not compact. If we choose a non-negative function f with compact support K satisfying $f \not\equiv 0$, $f < 1$, then clearly

$$\int f d\mu < \int 1 d\mu = \mu(G)$$

Let s_1 be the identity e . We may find $s_2 \in (KK^{-1})^c$ so that s_2K is disjoint from K . Then we still have $f + s_2f \leq 1$, but

$$\int f + s_1 f d\mu = \int f d\mu + \int s_2 f d\mu = 2 \int f d\mu < \mu(G)$$

We may repeat this process, finding

$$s_{n+1} \in \left(\bigcap_{i=1}^n s_i K K^{-1} \right)^c$$

so that $s_{n+1}K$ is disjoint from all of the previous s_iK . Such an s_{n+1} always exists because $\bigcup_{i=1}^n s_i K K^{-1}$ is a finite union of compact sets, hence is compact and may not cover G . Then for any n we have

$$\int \sum_{i=1}^n s_i f d\mu = n \int f d\mu < m(G)$$

which implies $m(G) = \infty$. □

The next proposition considers the particularly nice case of compact groups, which come with a nearly unique (after normalizing $\mu(G) = 1$) measure that is both left and right invariant. To help with the proof, we introduce the idea of a modular function, which give a way of creating a right Haar measure given a left invariant measure (or vice versa).

Proposition 9. *Let G be compact and μ a left-invariant measure on G , ν a right-invariant measure. Then μ and ν coincide, and moreover, if we normalize $\mu(G) = 1$, μ is unique.*

Proof. Let μ be a left-invariant measure on G . Denote by $\mu(f) = \int f d\mu$ the linear functional for $f \in \mathcal{K}(G)$. For any $t \in G$, consider the new functional where we right-multiply by t :

$$\mu_t(f) = \int f(xt^{-1})d\mu$$

It is easy to check that μ_t is once again a left Haar measure. Linearity properties of the functional are almost immediate, and we check left-invariance:

$$\mu_t(sf) = \int f(s^{-1}xt^{-1})d\mu = \int f(xt^{-1})d\mu = \mu_t(f)$$

This holds for any $f \in \mathcal{K}(G)$, so by uniqueness of the left Haar measure, there is a positive constant $\Delta(t)$ such that $\mu_t(f) = \Delta(t)\mu(f)$, and we may think of $\Delta : G \rightarrow (0, \infty)$. We note that

$$\Delta(st)\mu(f) = \int f(xt^{-1}s^{-1})d\mu = \Delta(s) \int f(xt^{-1})d\mu$$

$$= \Delta(s)\Delta(t) \int f d\mu = \Delta(s)\Delta(t)\mu(f)$$

So $\Delta(st) = \Delta(s)\Delta(t)$. Furthermore, a simple argument (see Cohn) shows that for any topological group G and regular Borel measure μ , maps of the form

$$x \mapsto \int f(yx) d\mu$$

are continuous. Choosing $f \neq 0$ and using $\mu_t(f) = \Delta(t)\mu(f)$ from above, we conclude that $\Delta : G \rightarrow [0, \infty)$ is continuous.

If G is compact, then, the image $\Delta(G)$ is bounded. But if for any $t \in G$, $\Delta(t) = k > 1$, then $\Delta(t^n) = k^n$ is unbounded as we take $n \rightarrow \infty$. If $\Delta(t) < 1$, then we can easily verify that $\Delta(t^{-1}) > 1$, so the same argument applies. We conclude that $\Delta \equiv 1$ when G is compact. Then to recap, we have

$$\mu(f) = \int f(x) d\mu(x) = \delta(t) \int f(xt^{-1}) d\mu(x) = \int f(xt^{-1}) d\mu(x)$$

so that μ is actually right-invariant as well! So if we divide the measure by $\mu(G)$, we have a unique left and right Haar measure when G is compact. \square

3 Examples

In this section, we will consider some locally compact groups and their Haar measures. Typically, instead of following the construction of the last section, we find that we may make a clever choice of defining a left-invariant integral on the set of continuous functions with compact support $\mathcal{K}(G)$. Theorem 2 assures us that such an integral must exist, and then we may use Theorem 1 to conclude that there is a corresponding left Haar measure.

First, we recall the example of the Lebesgue measure:

Example 2. The Lebesgue measure λ on \mathbb{R} that assigns each interval its length $\lambda([a, b]) = b - a$ is a Haar measure. Since $(\mathbb{R}, +)$ is abelian, λ is the only shift-invariant measure up to a scalar factor on \mathbb{R} defined on the Borel sets.

In light of the above discussion, we could arrive at the Lebesgue measure simply by taking the (relatively simple) notion of Riemann integration on continuous functions with compact support. The Riesz representation theorem then tells us the corresponding corresponding Borel measure exists, and Theorem 2 tells us this is the only shift-invariant measure.

This is perhaps slightly less powerful than it might seem in the previous sentence. We are effectively letting the Riesz representation theorem do most of the measure theoretic work that normally takes up the construction of the Lebesgue measure. We should also note that this theorem only tells us that the Lebesgue measure is defined for Borel sets; we would need to do additional work to extend it to the strictly larger class of Lebesgue measurable sets. This larger class of measurable sets is necessary, for example, to get the full benefits of the Lebesgue integral.

We now consider the group $\mathbb{R}^+ = (\mathbb{R}, \cdot)$, the positive reals under multiplication:

Example 3. Here, it is not as obvious that an invariant measure on \mathbb{R}^+ exists: under such a measure, the intervals $[1, 2]$ and $[2, 4]$ should have the same length, so the Lebesgue measure clearly does not work. But Theorem 2 assures us that the measure does exist, and after some thought we might note that

$$\int_1^2 \frac{1}{x} dx = \log(2) - \log(1) = \log \frac{2}{1} = \log \frac{4}{2} = \log 4 - \log 2 = \int_2^4 \frac{1}{x} dx$$

For a continuous function with $f : (0, \infty) \rightarrow \mathbb{R}$ with support $[a, b]$, then a simple substitution shows:

$$\int_a^b \frac{f(x)}{x} dx = \int_{ta}^{tb} \frac{f\left(\frac{x}{t}\right)}{\frac{x}{t}} \cdot \frac{dx}{t} = \int_{ta}^{tb} \frac{f\left(\frac{x}{t}\right)}{x} dx$$

So that integration with respect to $\frac{dx}{x}$ gives is left-invariant, where in this case, left-multiplying by t gives $t \cdot f(x) = f(x/t)$. So we may think of $m = \frac{dx}{x}$ as the Haar measure on \mathbb{R}^+ , where we understand that to get $m(O)$ for a Borel set O , we integrate its characteristic function $\int \chi_O \frac{dx}{x}$. We note that we may also view this as a Riemann-Stieltjes integral with integrating factor $d \log x$.

4 Sources

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