Haar Measures on Compact Groups

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November 23 2020

1 Introduction

This article provides a concise introduction to the theory of Haar measures on locally compact Hausdorff groups. We will cover the necessary preliminaries on topological groups and measure theory, the Haar measure, and unimodularity of compact groups. Haar measure will help us build useful representations and useful metrics on locally compact groups. When the group is compact, its representation into a collection of finite-dimensional representations, leading to the Peter–Weyl theorem which is used for understanding the structure of compact groups. In the compact abelian case, it turns out that the representations can be decomposed into one-dimension, which leads to the theory of Fourier analysis. Haar measures also show up in various applications. The concentration of Haar measures arises in the discussion of random matrices and random permutations. In Bayesian statistics, Haar measures are used for prior measures to formulate prior probabilities for compact groups of transformations.

2 Definitions

In the following sections, we follow the notations of [1] and [2].

2.1 Topology

A *group* is a set G with the multiplication map $m : G \times G \to G$ such that, denoting $m(x, y)$ with xy for all $x, y \in G$, we have

(i) $(xy)z = x(yz)$ for all $x, y, z \in G$.

(ii) $\exists e \in G$ such that $xe = ex = x$ for all $x \in G$.

(iii) $\forall x \in G \exists x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e$.

A topological group is a group together with a Hausdorff topology such that the maps

$$\begin{aligned} G \times G \to G \quad & (x, y) \mapsto xy \\ G \to G \quad & x \mapsto x^{-1} \end{aligned}$$

are continuous. As a consequence, left and right multiplication by elements of G as well as inversion are homeomorphisms of G.

A topological space is locally compact if every point has a compact neighbourhood, and it is Hausdorff when any two distinct points have disjoint neighbourhoods. In the Hausdorff case, local compactness is equivalent to every point having an open neighbourhood with compact closure. We note that the class of locally compact Hausdorff groups is stable under taking closed subgroups. If X is a topological space and $A \subset X$, we can equip A with the subspace topology, for which $U \in A$ is open if and only if there is an open set $V \subset X$ such that $U = A \cap V$.

Proposition 2.1. If X is a locally compact Hausdorff space and $A \subset X$ is closed, then A is locally compact Hausdorff.

Proof. We recall that compact subsets of Hausdorff spaces are closed and that closed subsets of compact sets are compact. This is follows from the definitions. □

For coset spaces, we recall the following lemma on a property of neighbourhoods that comes with the group structure.

Lemma 2.1. Let G be a topological group. Then for all $x \in G$ and all neighbourhood U of $e \in G$, there exists an open neighbourhood V of x such that $V^{-1}V \subset U$.

Proof. Let $\phi : G \times G \to G$ be the continuous map $\phi(g, h) := g^{-1}h$. There exists open neighborhoods of x $V_1, V_2 \subset G$ such that $V_1^{-1}V_2 = \phi(V_1, V_2)$. We take $V := V_1 \cap V_2$ which meets the requirement. □

When G is a topological group and $H \leq G$ is a subgroup of G, we equip the set of cosets G/H with the quotient topology. That is, $U \in G/H$ is open if and only if $\pi^{-1}(U) \subset G$ is open, with $\pi : G \to G/H$ defined as $\pi(g) := gH$. Then π is continuous and open, and left multiplication by $g \in G$ is a homeomorphism of G/H.

Proposition 2.2. Let G be a topological group and let $H \leq G$ be a closed subgroup. Then G/H is Hausdorff.

Proof. Let $xH, yH \in G/H$ be two distinct cosets. Then $yHx^{-1} \subset G$ is closed and does not contain $e \in G$. By Lemma 2.1, there is an open neighbourhood $V \subset G$ of $e \in G$ with $V^{-1}V \subset G \setminus yHx^{-1}$. Then VxH and VyH are disjoint neighbourhoods of xH and yH respectively. □

Proposition 2.3. Let G be a topological group and let $H \leq G$ be a subgroup. Then G/H is locally compact.

Proof. It suffices to show that $H \in G/H$ has a compact neighbourhood. Since G is locally compact, there is a compact neighbourhood U of $e \in G$. By Lemma 2.1, there exists an open neighbourhood V of e such that $V^{-1}V \subset U$. Then $\pi(V)$ is an open neighborhood of H since π is open. We show that $\overline{\pi(V)}$ is compact. If $gH \in \overline{\pi(V)}$, then $VgH \cap VH \neq \emptyset$, and $gH = v^{-1}uH$ for some $v, u \in V$. Hence $\overline{\pi(V)} \subset \pi(U)$, which is compact since π is continuous. □

We now state a version of Urysohn's Lemma which guarantees the existence of certain compactly supported functions on locally compact Hausdorff spaces. We recall that when X is a topological space, $C_c(X)$ denotes the set of continuous, complex-valued functions f on X with compact support

$$\mathrm{supp}(f) := \overline{\{x \in X | f(x) \neq 0\}}$$

When $f \in C_c(X)$ is such that $0 \leq f(x) \leq 1$ for all $x \in X$, $U \subset X$ is open, and $K \subset X$ is compact, we write $f \prec U$ if $\mathrm{supp}(f) \subset U$, and $K \prec f$ if $f(K) = 1$ for all $k \in K$.

Lemma 2.2. Let X be a locally compact Hausdorff space. When $K \subset X$ is compact and $U \subset X$ is open with $K \subset U$, there exists an open set $V \subset X$ with compact closure such that $K \subset V \subset \overline{V} \subset X$.

Lemma 2.3. (Urysohn) Let X be a locally compact Hausdorff space. When $K \subset X$ is compact and $U \subset X$ is open such that $K \subset U$, then there exists $f \in C_c(G)$ satisfying $K \prec f \prec U$.

Proposition 2.4. Let G be a locally compact Hausdorff group. Then any $f \in C_c(G)$ is uniformly continuous on the left and right. That is, for all $\epsilon > 0$ there is an open neighbourhood U of $e \in G$ such that for all $x \in G$ and $g \in U$ we have $|f(gx) - f(x)| < \epsilon$ and $|f(xg) - f(x)| < \epsilon$.

2.2 Measure theory

We now review some basic measure theory in order to give the definition of a Haar measure and some basic properties.
Let X be a non-empty set. A *σ-algebra* on X is a collection $\mathcal{M} \in \mathcal{P}(X)$ which is closed under complements and countable unions, and contains the empty set. A pair $(X, \mathcal{M})$ where X is a set and $\mathcal{M}$ is a σ-algebra on X is a *measurable space*, where any subset of $\mathcal{M}$ is *measurable*.

Given two measure spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, a map $f : X \to Y$ is *measurable* if $f^{-1}(S) \in \mathcal{M}$ for all $f(S) \in \mathcal{N}$. For example, let X and Y be topological spaces equipped with their Borel σ-algebras $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, i.e. the smallest σ-algebra containing all the open sets. Then any continuous map from X to Y is measurable. We will naturally equip topological spaces with their Borel σ-algebra.
A *measure* on a measurable space $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $\mu(\emptyset) = 0$ and that it is countably additive: if $\{S_i\}_{i \geq 1}$ is a countable collection of pairwise dijoint elements of $\mathcal{M}$, then $\mu\left(\cup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} \mu(S_i)$
The triplet $(X, \mathcal{M}, \mu)$ forms a *measure space*, and the elements of $\mathcal{M}$ are *measurable sets*. The definition of measure spaces is designed to allow for the following notion of integration of certain measurable, complex-valued functions on $(X, \mathcal{M}, \mu)$.

1. Let $\mathbf{1}_S$ be the characteristic function of a measurable set $S \in \mathcal{M}$. Define
$$\int_X \mathbf{1}_S(x)\mu(x) = \mu(S)$$

2. If $f := \sum_{i=1}^{n} \alpha_i \mathbf{1}_{S_i}$ is a positive real linear combination of characteristic functions of measurable sets, then we call f a *simple function*, and write
$$\int_X f(x)\mu(x) = \sum_{i=1}^{n} \alpha_i \mu(S_i)$$

3. If $f : X \to \mathbb{R}$ is measurable and non-negative, we define
$$\int_X f(x)\mu(x) = \sup_h \int_X h(x)\mu(x)$$
where h is any simple function on X with $0 < h < f$.

4. If $f : X \to \mathbb{R}$ is measurable, we can decompose it by $f = f_+ - f_-$ where $f_\pm := \max(\pm f, 0)$. If $\int_X |f(x)|\mu(x) < \infty$, we say that f is *integrable*, and we define
$$\int_X f(x)\mu(x) = \int_X f_+(x)\mu(x) - \int_X f_-(x)\mu(x)$$

5. If $f : X \to \mathbb{C}$ is measurable and integrable, we define
$$\int_X f(x)\mu(x) = \int_X \mathrm{Re}(f(x))\mu(x) + i \int_X \mathrm{Im}(f(x))\mu(x)$$

The vector space of equivalence classes of measurable and integrable complex-valued functions on X modulo equality on a conull set is denoted by $L^1(X, \mu)$. Integration constitutes a linear map from $L^1(X, \mu)$ to $\mathbb{C}$.
We will introduce Fubini's Theorem to reduce integrating over a product space to integrating over the factors. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Then we know that $(X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu)$, where $(\mu \times \nu)(S \times T) := \mu(S)\nu(T)$ for all $(S, T) \in \mathcal{M} \times \mathcal{N}$.

Theorem 2.1. (Fubini) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces (both X and Y are countable unions of sets of finite measure). If $f : X \times Y \to \mathbb{C}$ is measurable and integrable, then
$$\int_X \int_Y f(x, y)\nu(y)\mu(x) = \int_{X \times Y} f(x, y)(\nu \times \mu)(x, y) = \int_Y \int_X f(x, y)\mu(x)\nu(y)$$

Measures on topological spaces that appear in practice often satisfy the following additional regularity properties.

Definition 2.1. (Radon measure) A Radon measure on a topological space X is a measure μ on $(X, \mathcal{B}(X))$ such that

1. (locally finite) If $K \subset X$ is compact, then $\mu(K) < \infty$
2. (outer regular) If $S \subset X$ is measurable, then $\mu(S) = \inf\{\mu(U) | S \subset U, U \text{ open}\}$
3. (inner regular) If $U \subset X$ is open, then $\mu(U) = \sup\{\mu(K) | K \subset U, K \text{ compact}\}$

Theorem 2.2. (Riesz) Let X be a locally compact Hausdorff space. Let $\lambda : C_c(X) \to \mathbb{C}$ be a positive linear functional. That is, $\lambda(f) \geq 0$ if $f(x) \geq 0$ for all $x \in X$. Then there exists a unique Radon measure μ on X such that for all $f \in C_c(X)$, we have

$$\lambda(f) = \int_X f(x)\mu(x)$$

Moreover, μ statisfies $\mu(U) = \sup\{\lambda(f) | f \prec U, U \text{ open}\}$ and $\mu(K) = \inf\{\lambda(f) | K \prec f, K \text{ compact}\}$.

2.3 Haar measure

In the context of topological groups it is natural to look for measures which are invariant under translation. Such measures always exist for locally compact Hausdorff groups.

Definition 2.2. (Haar measure) Let G be a locally compact Hausdorff group. A *left (right) Haar measure* is a Radon measure μ on $(G, \mathcal{B}(G))$ which is non-trivial, and invariant under left-translation (right-translation). That is

1. (non-trivial) If $U \subset G$ is open and non-empty, then $\mu(G) > 0$
2. (translation-invariant) For all $S \in \mathcal{B}(G)$ and $g \in G$, we have $\mu(gS) = \mu(S)$ $(\mu(Sg) = \mu(S))$, where $gS = \{gs | s \in S\}$ $(Sg = \{sg | s \in S\})$.

Theorem 2.3. (Haar, Weil) If G is a locally compact group, then there exists a left (and right) Haar measure on G which is unique up to scalar multiple.

Example 1. Let G be a discrete group. Then $\mathcal{B}(G) = \mathcal{P}(G)$. The counting measure on G, defined by $\mu : \mathcal{P}(G) \to \mathbb{R}_{\geq} \cup \{\infty\}$ with $\mu(S) = |S|$, is a left and right Haar measure.

Due to Theorem 2.2, there exists a one-to-one correspondence between Haar measures and Haar functionals. We recall that a topological group G acts on $C_c(G)$ via the left-regular and the right-regular representations $L_g f(x) := f(g^{-1}x)$ and $R_g f(x) := f(xg)$ respectively, where $g, x \in G$ and $f \in C_c(G)$.

Definition 2.3. Let G be a locally compact Hausdorff group. A left (right) Haar functional on G is a non-trivial positive linear functional on $C_c(G)$ that is invariant under $L_g f(x)$ $(R_g f(x))$.

Proposition 2.5. Let G be a locally compact group, and let μ be a regular Borel measure on G which is finite on all compact subsets of G.

1. The measure μ is a left Haar measure on G if and only if the measure $\tilde{\mu}$, defined by $\tilde{\mu}(A) = \mu(A^{-1})$ for $A \in \mathcal{B}$, is a right Haar measure on G.
2. If μ is a left Haar measure on G, and ϕ is a continuous automorphism of G with continuous inverse, then $\mu \circ \phi$ is a left Haar measure on G.
3. The measure μ is a left Haar measure on G if and only if for every function $f \in C_c(G)$,
$$\int_G L_g f \, d\mu = \int_G f \, d\mu \quad \text{for all } g \in G$$
4. If μ is a left Haar measure on G, then μ is positive on all nonempty open subsets of G, and
$$\int_G f \, d\mu > 0 \quad \text{for all } f \in C_c(G)$$
5. If μ is a left Haar measure on G, then $\mu(G)$ is finite if and only if G is compact.

Proof. We will prove the last statement in 2.5. If G is compact, then $\mu(G) < \infty$ by Definition 2.1. Conversely, suppose that G is not compact. Let U be a relatively compact neighborhood of $e \in G$. Then there exists an infinite sequence (g_n), where $(g_n) \in G$ and $n \in \mathbb{N}$, such that $g_n \notin \cup_{k<n} g_k U$. Let V be defined as in Lemma 2.1. The sets $g_n V$ are pairwise disjoint since $VV^{-1} \subset U$. Since V has positive measure, G has infinite measure. □

3 Existence and Uniqueness

A Haar measure always exists for locally compact topological groups. We establish the following theorem for the existence and uniqueness of Haar measures.

Theorem 3.1. (Haar, Weil) If G is a locally compact group, then there exists a left (and right) Haar measure on G which is unique up to scalar multiple.

If $x \in G$, then the map $\phi_x : g \mapsto x^{-1}gx$ is a continuous automorphism of G with continuous inverse. If μ is a left Haar measure on G, then by part (2) of Proposition 2.5, $\mu \circ \phi_x$ is also a left Haar measure on G. By Theorem 3.1, there is some positive scalar $\delta_G(x)$ such that for every Borel set A of G, $(\mu \circ \phi_x)(A) = \delta_G(x)\mu(A)$. Note that $\delta_G(x)$ is independent of the initial choice of μ, since μ is unique up to scalar multiple.

Proposition 3.1. The function $\delta_G : G \to \mathbb{R}^{\times}_{>0}$ is a continuous homomorphism.

Proof. Let μ be a left Haar measure on G. For all $g, h \in G$, we have

$$\delta_G(gh)\mu = \mu \circ \phi_{gh} = \delta_G(h)\delta_G(g)\mu = \delta_G(g)\delta_G(h)\mu$$

Evaluating on a set of non-zero finite measure proves that indeed $\delta_G(gh) = \delta_G(g)\delta_G(h)$. It suffices to check continuity at $e \in G$ since δ_G is a homomorphism. Let F be the left Haar functional associated to μ, and let K be a compact neighborhood of e. Using Urysohn's Lemma 2.3, we choose $f_1, f_2 \in C_c(G)$ such that $K \prec f_1 \prec G$ and $K\text{supp}(f_2) \prec f_2 \prec G$. We recall that f_1 is uniformly continuous on the right. Given any $\epsilon > 0$, we can find a symmetric open neighbourhood $U \subset K$ around e such that $|f_1(xg) - f_1(x)|$ for all $g \in U$. Then we have

$$|\delta_G - \mathbf{1}_G| = \frac{1}{F(f_1)}|\delta_G F(f_1) - F(f_1)| \leq \frac{1}{F(f_1)} F(|R_g f_1 - f_1| f_2) \leq \epsilon \frac{f_2}{F(f_1)}$$

□

We let μ_L denote the left Haar measure on G. Since δ_G is continuous, the map $f \mapsto \int_G f(g)\, \delta_G(g) d\mu_L(g)$ is a positive linear functional on $C_c(G)$, and so by Theorem 2.2, it corresponds to a regular Borel measure on G which is positive on compact sets.

Proposition 3.2. Let μ_L be a left Haar measure on G. Then for any measurable function $F : G \to \mathbb{R}_{\geq 0}$, we have $\int_G F \circ \phi_x \, d\mu_L = \delta_G(x) \int_G F \, d\mu_L$. Also, the measure corresponding to the positive linear functional $f \mapsto \int_G f(g)\, \delta_G(g) d\mu_L(g)$ is a right Haar measure on G.

The function δ_G is the *modular quasicharacter* of the locally compact group G. If μ_L is a left Haar measure for G, then for any $x \in G$ and measurable set A, we have $\mu_L(Ax) = \mu_L(x^{-1}Ax) = \delta_G(x)\mu_L(A)$. That is, μ_L is bi-invariant, and thus a right Haar measure, if and only if $\delta_G(x) = 1$ for every $x \in G$. A locally compact group for which every left Haar measure is also a right Haar measure is called a *unimodular* group. It is immediate that abelian groups are unimodular.

Proposition 3.3. Every locally compact Hausdorff group G is unimodular. That is, every left Haar measure on G is also a right Haar measure on G and conversely.

4 Examples

We will offer some familar examples of locally campact groups and their Haar measures.

Example 2. On $G := (\mathbb{R}, +)$, a left and right Haar measure is given by the Lebesgue measure μ, which can be defined as the Radon measure associated with the Riemann integral $\int_{\mathbb{R}} : C_c(\mathbb{R}) \to \mathbb{C}$.

Proof. We can see that μ is left-translation invariant. Let $c \in \mathbb{R}$ be a constant. Then for any $f \in C_c(\mathbb{R})$ we can apply the change of variables

$$\int_{\mathbb{R}} f(c+x)\mu(x) = \int_{\mathbb{R}} f(c+x)\mu(c+x) = \int_{\mathbb{R}} f(y)\mu(y)$$

□

Example 3. On $G := (\mathbb{R}^n, +), n \geq 1$, a left and right Haar measure is given by the nth power of the Lebesgue measure μ.

Example 4. On $G := (\mathbb{R}^*, \cdot)$, the Lebesgue measure is not translation-invariant. But a left and right Haar functional can be given by $F : C_c(G) \to \mathbb{C}$ with

$$F(f) = \int_{\mathbb{R}} f(x)\frac{\mu(x)}{|x|}$$

Proof. For simplicity, we only show that the Haar functional is left invariant.

$$F(L_g f) = \int_{\mathbb{R}} f(cx)\frac{\mu(x)}{|x|} = \int_{\mathbb{R}} f(cx)\frac{\mu(cx)}{|cx|} = \int_{\mathbb{R}} f(y)\frac{\mu(y)}{|y|} = F(f)$$

□

Example 5. On $G := \mathrm{GL}(n, \mathbb{R})$ where $n \geq 1$, a left and right Haar functional can be given by $F : C_c(G) \to \mathbb{C}$ with

$$F(f) = \int_{\mathbb{R}} f(x)\frac{\mu(x)}{|\det(X)|^n}$$

where $\mu(X) := \sum_{i,j=1}^{n} \mu(x_i j)$ is the Lebesgue measure on $\mathbb{R}^{n \cdot n}$ in which G is an open subset.

Proof. The integral is finite by compactness of $\mathrm{supp}(f)$. Invariance can be checked by changing variables. □

Example 6. We now provide an examples of non-unimodular group. Let $G \subset$ SL(2,R) be the group

$$\left\{ \begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix} | x \in \mathbb{R} \setminus 0, y \in \mathbb{R} \right\}$$

Then the left Haar functional is given by ${}_L : C_c(G) \to \mathbb{C}$ with

$${}_L(f) := \int_{\mathbb{R}^2} f(X)\frac{\mu(x)\mu(y)}{x^2}$$

And the right Haar functional is given by ${}_R : C_c(G) \to \mathbb{C}$ with

$${}_R(f) := \int_{\mathbb{R}^2} f(X)\mu(x)\mu(y)$$

Example 7. Consider the group $G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R}, x > 0 \right\}$, which is homeomorphic to the open set $\mathbb{R}_{>0} \times \mathbb{R}$ in $\mathbb{R}^2$. Let $t = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G$, and consider left multiplication on G by t. We derive the map $x \mapsto ax$ and $y \mapsto ay + b$ of $\mathbb{R}_{>0} \times \mathbb{R}$ onto itself, and we name the linear map T. Then the Jacobian is $|J_T| = a^2$. Let $f : G \to \mathbb{R}_{\geq 0}$ be any Lebesgue measurable function, so that $\theta(x, y) = f(x, y)x^{-2}$ is also Lebesgue measurable. By change of variables, we have

$$\int_G f(x, y)x^{-2}\, dx\, dy = \int_G \theta \circ T\, a^2\, dx\, dy = \int_G (f \circ T)(x, y)\, x^{-2}\, dx\, dy.$$

In other words, if $g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, then for any $t \in G$, we have

$$\int_G f(tg)\, x^{-2}\, dx\, dy = \int_G f(g)\, x^{-2}\, dx\, dy,$$

and thus $\int x^{-2}\, dx\, dy$ is the left Haar measure on G.
We can also consider right multiplication on G by t, which gives the linear map $S : x \mapsto ax, y \mapsto bx + y$, where $|J_S| = a$. The change of variables formula gives

$$\int_G f(gt)\, x^{-1}\, dx\, dy = \int_G f(g)\, x^{-1}\, dx\, dy,$$

so that $\int x^{-1}\, dx\, dy$ is the right Haar measure on G. We note that G is not unimodular, and by Proposition 3.2 the modular quasicharacter of G is given by $\delta_G(g) = x$.

References

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