Baire Spaces

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Abstract

This expository paper on Baire Spaces is a submission as the project for the course MATH-4051 Topology (Fall 2021) taught by Professor Mike Miller Eismeier. A prospective reader should have done a first course in topology and analysis; everything else is developed from scratch. I have freely borrowed and modified from all the references listed at the end.

Contents

The Baire category is a profound triviality which condenses the folk wisdom of a generation of ingenious mathematicians into a single statement.

Tom Körner [Körner, 2008]

1 INTRODUCTION

Every mathematician has a few tricks [Rota, 2008] which form the foundation of their toolkit that they apply skilfully to a great variety of problems [Tao, 2021]. One of the goal of this expository note to convince the reader that Baire category theorems are a worthwhile addition to their toolkit. The other goal is to simply indulge in the study of Baire spaces for it is beautiful mathematics.

1.1 Small Sets

What is a "small" set? This question can have multiple answers depending on who you ask. For a set theorist, a countable set is "small". For an analyst, a set of measure 0 is "small". We will see that for a topologist the ideas of nowhere dense set and meager set encapsulate the notion of "smallness". These notions don't always coincide though. For example, Cantor set is an uncountable set having Lebesgue measure 0, and as we will see in Section 2 it is also nowhere dense in $\mathbb{R}$.

Using this topological vocabulary for small sets we will define the notion of a Baire space, which in this vocabulary aren't "small". Many well-behaved classes of spaces are Baire. Among the most well-known are locally compact Hausdorff spaces and complete pseudometric spaces. This is the content of the famous Baire category theorems which will be the highlight of this article.

These Baire Category theorems are a powerful class of results useful for proving existence. We will see in Section 7 how they can be applied to prove existence of objects of desired property. The main idea which these applications follow is to first construct a set X in which we want to prove there exists elements satisfying a certain property, and then show that elements *not* satisfying this property, in fact, form a "small set".

1.2 Warm-Up

As a warm-up to the kind of arguments we will be dealing with let's show that

every nonempty compact Hausdorff space with no isolated points is uncountable.

This will in particular show that $[0, 1]$ is uncountable.

We first show that given any nonempty open set U of X and any point $x \in X$, there exists a nonempty open set V contained in U such that x is not contained in the closure $\mathrm{cl}_X(V)$ of V in X. If $x \in U$, then since $\{x\}$ is not open, U must contain a point different from x. On other hand, if $x \notin U$, then since U is nonempty it must contain a point different from x. In either case denote this point by y. X is Hausdorff, and therefore we can choose two disjoint open subsets W_1, W_2 of X such that $x \in W_1$ and $y \in W_2$. Then let $V = W_2 \cap U$. It is an open set contained in U, it is nonempty because it contains y, and its closure does not contain x.

To finish proving that X is uncountable we show that any given function $f : \mathbb{N} \to X$ is not surjective. Let $x_n = f(n)$. Apply our result above to the nonempty open set $U = X$ to choose a nonempty open subset V_1 of X such that $x_1 \notin \mathrm{cl}_X(V_1)$. In general, for each $n \in \mathbb{N}$, given nonempty open subset V_n of X, choose V_{n+1} using the result above such that $V_{n+1} \subseteq V_n$ and $x_{n+1} \notin \mathrm{cl}_X(V_{n+1})$. We get a nested sequence

$$\mathrm{cl}_X(V_1) \supseteq \mathrm{cl}_X(V_2) \supseteq \cdots$$

of nonempty closed subsets of X. Now recall the classical result that a topological space X is compact if and only if for every collection $\mathcal{C}$ of closed subsets of X having the finite intersection property (i.e., every finite sub-collection of $\mathcal{C}$ has nonempty intersection), the intersection $\bigcap_{C\in\mathcal{C}} C$ is nonempty (See Proposition 5.4 for a proof). Therefore, there exists a point $x \in \bigcap_n \mathrm{cl}_X(V_n)$. Now x cannot equal x_n for any $n \in \mathbb{N}$, since x is in $\mathrm{cl}_X(V_n)$ but x_n is not. Therefore, f is not surjective.

We showed that X cannot be written as a countable union of singletons. A compact Hausdorff space is T_1, i.e., for any $x \in X$, the singleton $\{x\}$ is closed. We will show later that the singletons $\{x\}$ in this case are what is called nowhere dense, and that X cannot be written as a countable union of nowhere dense sets. This is exactly the content of Baire category theorems, and therefore we could have concluded uncountability directly from these theorems.

2 PREREQUISITES

In this section we give a brief tour of the prerequisites from point-set topology, mainly for notational convenience. For a more comprehensive treatment see the excellent textbooks [Munkres, 2000, Willard, 2004, Engelking, 1989].

We start by recalling the notions of interior and closure which will be used in defining the notions of dense and nowhere dense sets in a topological space. As we mentioned before they provide a topological vocabulary to deal with "large" and "small" sets. Throughout we simply write X for a topological space $(X, \mathcal{T})$.

Definition 2.1 (Interior and Closure)**.** Given a subset S of a topological space X, the *interior of S in X*, denoted $\mathrm{int}_X(S)$, is defined as the union of all open sets contained in S, and the *closure of S in X*, denoted $\mathrm{cl}_X(S)$, is defined as the intersection of all closed sets containing S.

Note that $\mathrm{int}_X(S)$ is an open set in X and $\mathrm{cl}_X(S)$ is a closed set in X.

Proposition 2.2 (Equivalent characterization of closure)**.** *Given a subset S of a topological space X, $x \in \mathrm{cl}_X(S)$ if and only if every open subset U of X containing x intersects S.*

Proof. We show that $x \notin \mathrm{cl}_X(S)$ if and only if there exists an open subset U of X containing x that does not intersect S. If $x \notin \mathrm{cl}_X(S)$, the set $U = X \setminus \mathrm{cl}_X(S)$ is an open subset of X containing x that does not intersect S. Conversely, if there exists an open subset U of X containing x that does not intersect S, then $X \setminus U$ is a closed subset of X containing S. By definition of the closure, the set $X \setminus U$ must contain $\mathrm{cl}_X(S)$, implying $x \notin \mathrm{cl}_X(S)$. □

Proposition 2.3 (Complements of interior and closure)**.** *For any subset S of a topological space X, $X \setminus \mathrm{int}_X(S) = \mathrm{cl}_X(X \setminus S)$ and $X \setminus \mathrm{cl}_X(S) = \mathrm{int}_X(X \setminus S)$.*

Proof. This follows immediately from the definitions of closure and interior, and the De Morgan's laws:

$$X \setminus \mathrm{int}_X(S) = X \setminus \bigcup_{\substack{U \subseteq S \\ U \text{ open in } X}} U = \bigcap_{\substack{U \subseteq S \\ U \text{ open in } X}} (X \setminus U) = \bigcap_{\substack{X\setminus U \supseteq X \setminus S \\ X\setminus U \text{ closed in } X}} (X \setminus U) = \bigcap_{\substack{C \supseteq X \setminus S \\ C \text{ closed in } X}} C = \mathrm{cl}_X(X \setminus S)$$

$$X \setminus \mathrm{cl}_X(S) = X \setminus \bigcap_{\substack{C \supseteq S \\ C \text{ closed in } X}} C = \bigcup_{\substack{C \supseteq S \\ C \text{ closed in } X}} (X \setminus C) = \bigcup_{\substack{X\setminus C \subseteq X \setminus S \\ X\setminus C \text{ open in } X}} (X \setminus C) = \bigcup_{\substack{U \subseteq X \setminus S \\ U \text{ open in } X}} U = \mathrm{int}_X(X \setminus S)\,.$$

□

Definition 2.4 (Dense and nowhere dense)**.** A subset S of a topological space X is said to be *dense in X* if its closure equals X, i.e., $\mathrm{cl}_X(S) = X$. S is called *nowhere dense* or *rare in X* if its closure has empty interior, i.e., $\mathrm{int}_X(\mathrm{cl}_X(S)) = \varnothing$.

Nowhere dense sets give a precise meaning to "sparsely populated sets" or "sets with holes" (see the characterization in Proposition 2.7 (c) for a more concrete depiction of this). Nowhere dense is a strengthening of the condition "not dense" (every nowhere dense set is not dense, but the converse is false).

We immediately give equivalent characterizations of dense and nowhere dense sets which we will freely use.

Proposition 2.5 (Equivalent characterisations of dense sets). *Let S be a subset of the topological space X. Then S is dense in X if and only if S has nonempty intersection with every nonempty open subset of X.*

Proof. S is dense in X means $\mathrm{cl}_X(S) = X$, which using Proposition 2.3 can written as $\mathrm{int}_X(X \setminus S) = \varnothing$. By the definition of interior this means that the only open set disjoint from S is the empty set. $\square$

Proposition 2.6 (Dense subset of a dense set is itself dense). *Suppose X is a topological space and $T \subseteq S \subseteq X$ are subspaces with T dense in S and S dense in X. Then T is dense in X.*

Proof. Let U be a nonempty open subset of X. Then $U \cap S$ in nonempty by Proposition 2.5. Now note that $U \cap S$ is an open subset of S in the subspace topology, and therefore $(U \cap S) \cap T$ is also nonempty. But this implies that T is dense in X. $\square$

Proposition 2.7 (Equivalent characterisations of nowhere dense sets). *Let S be a subset of the topological space X. Then the following are equivalent:*

(a) S is nowhere dense in X.

(b) $X \setminus \mathrm{cl}_X(S)$ is dense in X.

(c) Every nonempty open subset U of X contains a nonempty open subset V that is disjoint from S.

(d) S is not dense in any nonempty open subset U of X.

Proof. $(a) \iff (b)$: S is nowhere dense in X means $\mathrm{int}_X(\mathrm{cl}_X(S)) = \varnothing$, which using Proposition 2.3 can written as $\mathrm{cl}_X(X \setminus \mathrm{cl}_X(S)) = X$, i.e., $X \setminus \mathrm{cl}_X(S)$ is dense in X.

$(b) \implies (c)$: If $S = \varnothing$ then (c) holds trivially, so suppose S is nonempty. Then Propositions 2.3 and 2.5 imply that every nonempty open subset U of X intersects $\mathrm{int}_X(X \setminus S)$. Note that $\mathrm{int}_X(X \setminus S)$ is an open set disjoint from S and thus we can let $V = U \cap \mathrm{int}_X(X \setminus S)$.

$(c) \implies (b)$: If every nonempty open subset U of X contains a nonempty open subset V that is disjoint from S, then every nonempty open subset U of X intersects the open set $\mathrm{int}_X(X \setminus S)$, which by Propositions 2.3 and 2.5 imply (b).

$(c) \iff (d)$: Follows immediately from Proposition 2.5.

$\square$

Example 2.8.

1. The rationals $\mathbb{Q}$ are dense in $\mathbb{R}$, and hence not nowhere dense in $\mathbb{R}$.
2. $\mathbb{R}$ is nowhere dense in $\mathbb{R}^n$ for $n > 1$.
3. The integers $\mathbb{Z}$ are nowhere dense in $\mathbb{R}$. However, $\mathbb{Z} \cup (a, b)$ for $a < b$ is not nowhere dense in $\mathbb{R}$ since it is dense in (a, b). $\mathbb{Z} \cup (a, b)$ is not dense in $\mathbb{R}$ either.
4. The collection $S = \{\frac{1}{n}\}_{n \in \mathbb{N}}$ is nowhere dense in $\mathbb{R}$, since its closure $\mathrm{cl}_{\mathbb{R}}(S) = S \cup \{0\}$ has empty interior in $\mathbb{R}$.
5. Suppose X is a T_1 space, i.e., for any $x \in X$, the singleton $\{x\}$ is closed, and that the singletons are not isolated, i.e., for any $x \in X$, the singleton $\{x\}$ is not open. Then singletons are nowhere dense. This is true, for example, if X is $\mathbb{R}^n$ for any $n \in \mathbb{N}$.
6. The Cantor set C is nowhere dense in $\mathbb{R}$ with its Euclidean topology. To see this recall that the Cantor set is constructed using the decreasing sequence $\{C_n\}_{n \geq 0}$ of closed sets defined as follows: $C_0 = [0, 1]$, $C_1 = [0, 1/3] \cup [2/3, 1]$, $C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$, and so on, where at each step to get C_{n+1} from C_n we remove the middle third open interval from each of the 2^n closed disjoint subsets of C_n. The Cantor set is the countable intersection $C = \bigcap_{n \in \mathbb{N}} C_n$. C is therefore nowhere dense since C_n contains no interval of length greater than $1/3^n$.

7. Suppose X is a topological space and $S \subseteq X$. The boundary of S in X, denoted $\text{bd}_X(S)$ is defined to be the closed set $\text{cl}_X(S) \cap \text{cl}_X(X \setminus S)$. The neighborhood of any point in $\text{bd}_X(S)$ meets both S and $X \setminus S$. Therefore, if S is open in X, $\text{bd}_X(S) \subseteq X \setminus S$, and thus $\text{int}_X(\text{bd}_X(S)) = \varnothing$. Similarly, if S is closed, $\text{bd}_X(S) \subseteq S$ and thus $\text{int}_X(\text{bd}_X(S)) = \varnothing$. In other words, boundaries of open or closed sets is nowhere dense, and also a closed set is nowhere dense if and only if it coincides with its boundary.

2.1 Properties Of Nowhere Dense Sets

We next define an ideal and a $\sigma-$ideal. This definition will help us view many notions of "smallness" like being nowhere dense, being meager (see Definition 3.1), being countable or having measure 0, as related.

Definition 2.9. An *ideal* on a set X is a collection of subsets of X containing $\varnothing$ and closed under arbitrary subsets and finite unions. If the collection is also closed under countable unions it is called a *$\sigma-$ideal.*

Proposition 2.10. *The collection of nowhere dense sets in a topological space X form an ideal.*

Proof. The fact that a subset of a nowhere dense set in X is also nowhere dense in X is obvious. To prove that a finite union of nowhere dense sets in X is also nowhere dense in X we need only show that for two nowhere dense sets S_1 and S_2 in X. Using the characterization in Proposition 2.7 (c), for each nonempty open subset U of X we can find nonempty open subsets U_1 and U_2 of X such that $U_1 \subseteq U \setminus S_1$ and $U_2 \subseteq U_1 \setminus S_2$. Hence $U_2 \subseteq U \setminus (S_1 \cup S_2)$, and therefore again by Proposition 2.7 (c), $S_1 \cup S_2$ is nowhere dense in X. □

The union of countably many nowhere dense sets may not be nowhere dense as the example of $\mathbb{Q}$ in $\mathbb{R}$ shows.

The next proposition will often be useful since we will often have open or dense subsets.

Proposition 2.11. *Let Y be a subspace of a topological space X, and let S be a subset of Y. If S is nowhere dense in Y, then S is nowhere dense in X. Conversely, if Y is open or dense in X and S is nowhere dense in X, then S is nowhere dense in Y.*

Proof. Suppose first that S is nowhere dense in Y. Let U be a nonempty open subset of X that intersects Y. Then $U \cap Y$ is open in Y by the definition of subspace topology. By Proposition 2.7 there exists a nonempty open subset V of Y which is contained in $U \cap Y$ and is disjoint from S. By the definition of subspace topology on Y, there exists a nonempty open subset W of X such that $V = W \cap Y$. This set is such that $W \cap S = \varnothing$ and can taken to be such that $W \subseteq U$ by taking its intersection with U if necessary. By using Proposition 2.7 again we see that S is dense in X.

Now suppose that Y is open in X and that S is nowhere dense in X. Let U be any open subset of Y, which, because Y is open in X, is then also open in X. Therefore, by Proposition 2.7 there exists a nonempty open subset V of X which is contained in U and is disjoint from S. This set V is open in Y also because $V = V \cap Y$. Therefore, by Proposition 2.7 again, S is nowhere dense in Y.

Finally suppose that Y is dense in X and that S is nowhere dense in X. Let U be any open subset of Y. Then there exists an open subset W of X such that $U = W \cap Y$. By Proposition 2.7 there exists a nonempty open subset V of X which is contained in W and is disjoint from S. We see that the set $V \cap Y$ is nonempty by noting that Y is dense in X and then using Proposition 2.5. It is also open in Y, contained in U, and is disjoint from S. By Proposition 2.7 then, S is nowhere dense in Y. □

How does nowhere density behave with respect to taking products? As the following theorem shows, the behaviour is simple under finite products. For infinite products an extra (intuitive) condition can cause the product to be nowhere dense. As an example of this condition, let $X_n = [0, 1]$ and $S_n = [0, 1/2]$ for each $n \in \mathbb{N}$. Then for no $n \in \mathbb{N}$, is S_n is nowhere dense in X_n, but $\prod_{n \in \mathbb{N}} S_n$ is nowhere dense in $\prod_{n \in \mathbb{N}} X_n$.

Proposition 2.12 (Products of nowhere dense sets)**.** *Let A be a set. For all $\alpha \in A$, let X_α be a topological space and let $S_\alpha \subseteq X_\alpha$.*

If A is finite, then $\prod_{\alpha \in A} S_\alpha$ is nowhere dense in $\prod_{\alpha \in A} X_\alpha$ if and only if there exists a $\beta \in A$ such that S_β is nowhere dense in X_β.

If A is infinite, then $\prod_{\alpha\in A} S_\alpha$ is nowhere dense in $\prod_{\alpha\in A} X_\alpha$ if and only if there exists a $\beta \in A$ such that S_β is nowhere dense in X_β or there exists infinitely many $\beta \in A$ such that for each such β, S_β is not dense in X_β.

Proof. We note that although the theorem is stated separately for A finite and A infinite, having only the second statement without the "If A is infinite" is sufficient because the "or" part is redundant if A is finite. For notational simplicity, let $X = \prod_{\alpha\in A} X_\alpha$ and $S = \prod_{\alpha\in A} S_\alpha$.

We start with proving the contrapositive of the if side. Suppose that for each $\alpha \in A$, $\mathrm{int}_{X_\alpha}(\mathrm{cl}_{X_\alpha}(S_\alpha)) \neq \varnothing$ and that $\mathrm{cl}_{X_\alpha}(S_\alpha) = X_\alpha$ for all but finitely many $\alpha \in A$. Then, using the definition of product topology, it is easily seen that

$$\mathrm{int}_X(\mathrm{cl}_X(S)) = \prod_{\alpha\in A} \mathrm{int}_{X_\alpha}(\mathrm{cl}_{X_\alpha}(S_\alpha)) \neq \varnothing,$$

thereby showing that $\prod_{\alpha\in A} S_\alpha$ is not nowhere dense in $\prod_{\alpha\in A} X_\alpha$.

For the other side, we split the analysis into two parts depending on which condition is true. If for some $\beta \in A$, S_β is nowhere dense in X_β, i.e., $\mathrm{int}_{X_\beta}\big(\mathrm{cl}_{X_\beta}(S_\beta)\big) = \varnothing$, then $\prod_{\alpha\in A} S_\alpha$ is nowhere dense in $\prod_{\alpha\in A} X_\alpha$ since

$$\mathrm{int}_X(\mathrm{cl}_X(S)) = \prod_{\alpha\in A} \mathrm{int}_{X_\alpha}(\mathrm{cl}_{X_\alpha}(S_\alpha)) = \varnothing.$$

On the other hand, if $\mathrm{cl}_{X_\beta}(S_\beta) \neq X_\beta$ for infinitely many $\beta \in A$, and U is any element from the canonical basis of the product topology on X, then there exists a $\gamma \in A$ such that

$$\pi_\gamma(U) = X_\gamma \text{ and } \mathrm{cl}_{X_\gamma}(S_\gamma) \neq X_\gamma,$$

where $\pi_\gamma \colon X \to X_\gamma$ denotes the canonical projection. Define the nonempty open subset $V \subseteq U$ by

$$\pi_\gamma(V) = X_\gamma \setminus \mathrm{cl}_{X_\gamma}(S_\gamma) \text{ and } \pi_\alpha(V) = \pi_\alpha(U).$$

Then $V \cap S = \varnothing$, showing that S is nowhere dense in X. □

3 MEAGER SETS

Definition 3.1. A subset S of a topological space X is called *meager* or of *first category in X* if it can be written as a countable union of nowhere dense sets in X. Otherwise, S is called *nonmeager* or of *second category in X*.

We will eschew the usage of the terminology "first category" or "second category" because aside from being non-descriptive, the word "category" in mathematics has "been been conscripted for higher service" as S. Berberain remarked.

Remark 3.2. By Proposition 2.11, if Y is an open or dense subspace of X and $S \subseteq Y$, then S is meager (resp. nonmeager) relative to Y if and only if S is meager (resp. nonmeager) relative to X.

Example 3.3.

1. Rationals $\mathbb{Q}$ and integers $\mathbb{Z}$ are meager in $\mathbb{R}$ since they can written as a countable union of singletons which are nowhere dense sets in $\mathbb{R}$.
2. Every countable set is meager under the setting of Example 2.8.5.
3. The Cantor set is nowhere dense, and hence also meager. In fact, every nowhere dense in meager.
4. Any topological space which contains an isolated point t is nonmeager, as no set to which t belongs can be nowhere dense.
5. We will show in Section 7.3 that the set of functions that have a derivative at some point is a meagre set in the space of real-valued continuous functions on $[0, 1]$ with the uniform topology.

3.1 Properties Of Meager Sets

Proposition 3.4. *The collection of meager sets in a topological space X form a $\sigma-$ideal.*

Proof. A subset of a meager set in X is again meager in X because a subset of a nowhere dense set in X is also nowhere dense in X. A countable union of meager sets in X is again meager in X follows from the fact that the Cartesian product of two countable sets is also countable. □

It is easily seen that the class of $\mu-$null sets in a complete measure space $(X, \mathcal{X}, \mu)$ form a $\sigma-$ideal, and so does the class of countable subsets of any set X. Therefore, being a $\sigma-$ideal is a characteristic property of many notions of "smallness" of sets. With that said, these notions of "smallness" may of very different nature as the following theorem shows.

Theorem 3.5. *Endow the real line $\mathbb{R}$ with the Lebesgue measure λ. Then $\mathbb{R}$ can be written as $\mathbb{R} = A \cup B$, where $A \cap B = \varnothing$, A is meager in $\mathbb{R}$, and $\lambda(B) = 0$.*

Proof. Suppose $\{r_n\}_{n\in\mathbb{N}}$ in an enumeration of the rationals. Consider the family $\{I_{n,m}\}_{n,m\in\mathbb{N}}$ of open intervals defined by

$$I_{n,m} = \left(r_n - \frac{1}{2^{n+m+1}}, r_n + \frac{1}{2^{n+m+1}}\right),$$

i.e., $I_{n,m}$ is an open interval centered at r_n and has Lebesgue measure $1/2^{n+m}$. Also define $G_m = \bigcup_{n\in\mathbb{N}} I_{n,m}$ and $B = \bigcap_{m\in\mathbb{N}} G_m$.

We claim that $\lambda(B) = 0$. To this end, let $\varepsilon > 0$ be arbitrary. Let $M \in \mathbb{N}$ be large enough so that $1/2^M < \varepsilon$. Then since $B \subseteq G_M$ and since

$$\lambda(G_M) \leq \sum_{n\in\mathbb{N}} \lambda(I_{n,M}) = \sum_{n\in\mathbb{N}} \frac{1}{2^{n+M}} = \frac{1}{2^M} < \varepsilon,$$

we have $\lambda(B) < \varepsilon$.

Next, we show that

$$A = \mathbb{R} \setminus B = \bigcup_{m\in\mathbb{N}} (X \setminus G_m)$$

is meager in $\mathbb{R}$ by showing that $X \setminus G_m$ is nowhere dense in $\mathbb{R}$ for each $m \in \mathbb{N}$. But this is obvious from the definition of G_m and from the fact that rationals are dense in $\mathbb{R}$. □

It follows from this theorem that every subset of $\mathbb{R}$ can be partitioned into a meager set and a set of Lebesgue measure 0.

We next prove the Banach category theorem which says that any topological space is "almost" a Baire space (see Section 4 for the definition of a Baire space). More specifically, given this theorem, we can write any topological space X as a union of a Baire space (union of all nonmeager open sets) and a meager set (union of all meager open sets) since in a Baire space every nonempty open set is nonmeager.

Theorem 3.6 (Banach Category Theorem)**.** *In a topological space X, the union of any family of meager open sets is also meager.*

Proof. Let $\mathcal{U}$ be a family of nonempty meager open subsets of X, and denote by $O = \bigcup_{U\in\mathcal{U}} U$ its union. We want to show that O is meager in X.

Let $\mathscr{U}$ denote the set of all collections of pairwise disjoint nonempty open sets in X with the property that each member of each collection is a subset of some member of $\mathcal{U}$. We will use Zorn's lemma to find a maximal element of $\mathscr{U}$. $\mathscr{U}$ is partially ordered by set inclusion. Let $\mathscr{C}$ be a chain in $\mathscr{U}$. To apply Zorn's lemma, we need to show that $\mathcal{B} = \bigcup_{\mathcal{R}\in\mathscr{C}} \mathcal{R}$ is an upper bound of $\mathscr{C}$ in $\mathscr{U}$, i.e., we need to show $\mathcal{B} \in \mathscr{U}$ and $\mathcal{R} \subseteq \mathcal{B}$ for every $\mathcal{R} \in \mathscr{C}$. But both these claims are immediate from our construction. Therefore, by Zorn's lemma, $\mathscr{U}$ has a maximal element, say $\mathcal{V} = \{V_\alpha \ : \ \alpha \in A\}$ for some index set A. Denote by $V = \bigcup_{\alpha \in A} V_\alpha$ its union.

We next claim that $\mathrm{cl}_X(O) \setminus V$ is nowhere dense in X. Suppose to the contrary that this is not the case, and therefore by Proposition 2.7 (c), there exists a nonempty open subset W of X every nonempty open subset of which intersects $\mathrm{cl}_X(O) \setminus V$. In particular, W intersects $\mathrm{cl}_X(O) \setminus V$, which implies $W \cap O \neq \varnothing$. But then $W \cap U$ for some $U \in \mathcal{U}$ is a nonempty open set disjoint from all sets in $\mathcal{V}$, contradicting the fact that $\mathcal{V}$ is maximal.

This fact and the fact that $\mathcal{U}$ is a family of meager sets, means that V_α is meager for each $\alpha \in A$. Write $V_\alpha = \bigcup_{n \in \mathbb{N}} S_{\alpha,n}$ for nowhere dense sets $S_{n,\alpha}$. Define $S_n = \bigcup_{\alpha \in A} S_{\alpha,n}$. We now claim that S_n is nowhere dense for each $n \in \mathbb{N}$. To see this suppose W is a nonempty open set in X which intersects S_n. Then W intersects $S_{\alpha,n}$ for some $\alpha \in A$, and since $S_{\alpha,n}$ is nowhere dense, there exists a nonempty open set $W' \subseteq (W \cap V_\alpha) \setminus S_{\alpha,n}$. This gives $W' \subseteq W \setminus S_n$, and thus S_n is nowhere dense.

Finally, note that

$$O \subseteq (\mathrm{cl}_X(O) \setminus V) \cup V = (\mathrm{cl}_X(O) \setminus V) \cup \bigcup_{n \in \mathbb{N}} S_n,$$

or in other words, O is a subset of a countable union of nowhere dense sets, which is to say, O is meager. □

4 BAIRE SPACES

Definition 4.1. A topological space X is said to be a *Baire space* if every countable union of closed nowhere dense sets in X has empty interior in X.

There are many equivalent formulations for characterizing Baire spaces. Each formulation listed in the next Proposition is more convenient than the other formulations in some proofs, and therefore we will freely use any of them as the definition of a Baire space.

Proposition 4.2. *The following conditions on a topological space X are equivalent:*

(a) X is a Baire space.

(b) The intersection of countably many dense open sets in X is dense in X.

(c) Each nonempty open subset of X is nonmeager in X.

(d) Every meager set in X has empty interior in X.

(e) Complements of meager sets in X are dense in X.

Proof. (a) $\iff$ (b): If C is a closed nowhere dense subset of X, then by Proposition 2.3 $X \setminus C$ is dense in X. Therefore, taking complement and using Proposition 2.3 we see that countable union of closed nowhere dense sets in X has empty interior if and only if the intersection of countably many dense open sets is dense.

(c) $\implies$ (a): We will prove the contrapositive, i.e., if there exists a nonempty open subset of X which is meager in X, then there exists a countable collection of closed nowhere dense sets in X such that the interior of their union is nonempty. To that end, let $O = \bigcup_{n \in \mathbb{N}} S_n$ be a nonempty open subset of X which is meager in X, written as a countable union of nowhere dense sets $\{S_n\}_{n \in \mathbb{N}}$. Since closure of closure of a set in simply the closure, the collection $\{\mathrm{cl}_X(S_n)\}_{n \in \mathbb{N}}$ is a countable collection of closed nowhere dense sets in X. Therefore, the fact that O is nonempty and the following observation finishes the proof:

$$O = \mathrm{int}_X(O) = \mathrm{int}_X\left(\bigcup_{n \in \mathbb{N}} S_n\right) \subseteq \mathrm{int}_X\left(\bigcup_{n \in \mathbb{N}} \mathrm{cl}_X(S_n)\right).$$

(a) $\implies$ (c) We will again prove the contrapositive, i.e., if there exists a countable collection $\{F_n\}_{n \in \mathbb{N}}$ of closed nowhere dense sets in X such that the interior of their union is nonempty, then there exists a nonempty open subset of X which is meager in X. Denote the nonempty interior $\text{int}_X(\bigcup_n F_n)$ by O. Consider the countable collection $\{F_n \cap O\}_{n\in\mathbb{N}}$ and note that since a subset of a nowhere dense set is also nowhere dense, the collection consists of nowhere dense sets in X. The set O is a subset of $\bigcup_n F_n$ and therefore

$$O = O \cap \bigcup_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} (F_n \cap O),$$

showing that O is meager in X.

$(c) \iff (d)$: Follows immediately from the observation that a subset of a meager set in X is also meager in X.

$(d) \iff (e)$: Follows immediately from Proposition 2.3. □

Example 4.3.
1. Trivially, singleton sets are always Baire spaces.
2. Baire category theorems, proved in Section 5 provide the most common examples of Baire spaces, namely, locally compact Hausdorff spaces and complete pseudo-metric spaces. In particular, the Euclidean spaces $\mathbb{R}^n$ for all $n \in \mathbb{N}$ are Baire spaces.
3. The space $\mathbb{Q}$ of rationals (with the subspace topology as a subspace of $\mathbb{R}$) is not a Baire space. Indeed, singletons in $\mathbb{Q}$ are closed and have empty interior, and therefore $\mathbb{Q}$, which can be written as a countable union of all its singletons, cannot be Baire.
4. The space $\mathbb{R} \setminus \mathbb{Q}$ of irrationals (with the subspace topology as a subspace of $\mathbb{R}$) is a Baire space.
5. The space $\mathbb{N}$ of natural numbers (with the subspace topology as a subspace of $\mathbb{R}$) is a Baire space.

4.1 Properties Of Baire Spaces

Consider the Baire space $\mathbb{R}^2 \setminus ((\mathbb{R} \setminus \mathbb{Q}) \times \{0\})$. Its closed subset $\mathbb{Q} \times \{0\}$ is clearly meager. Therefore, a subspace of a Baire space may not be Baire. But if the subspace is open then it is Baire.

Proposition 4.4 (Open Subspaces)**.** *If X is a Baire space and $Y \subseteq X$ is open, then Y is also a Baire space.*

Proof. Let $\{S_n\}_{n\in\mathbb{N}}$ be a countable collection of closed nowhere dense sets in Y. We need to show that $\bigcup_{n\in\mathbb{N}} S_n$ has empty interior in Y. Since S_n is closed in Y, we have $\text{cl}_X(S_n) \cap Y = S_n$ for each $n \in \mathbb{N}$. We claim that $\text{cl}_X(S_n)$ is nowhere dense in X for each $n \in \mathbb{N}$. To see this, suppose to the contrary that there exists a nonempty open subset U of X such that $U \subseteq \text{cl}_X(S_n)$. Then U must intersect S_n. Therefore, $U \cap Y$ is a nonempty open subset of Y contained in S_n. This contradicts the fact that S_n is nowhere dense in Y.

We can now use the fact that X is a Baire space and get that $\bigcup_{n\in\mathbb{N}} \text{cl}_X(S_n)$ has empty interior. Now note that if $\bigcup_{n\in\mathbb{N}} S_n$ didn't have empty interior in Y, then there would exist a nonempty open subset V of Y such that $V \subseteq \bigcup_{n\in\mathbb{N}} S_n$. This set V is also open in X because Y is an open subset of X, and then $V \subseteq \bigcup_{n\in\mathbb{N}} \text{cl}_X(S_n)$ would give a contradiction.

For another proof recall Remark 3.2 and the characterization from Proposition 4.2 (c) of Baire spaces, and then use the fact that an open subset of Y is also an open subset of X. □

Even a dense subspace of a Baire space may not be Baire as is verified by the fact that $\mathbb{Q}$ is not a Baire space but is a dense subspace of the Baire space $\mathbb{R}$. However, a dense $G_\delta-$subspace of a Baire space is a Baire space. See Proposition 1.23 in [Haworth and McCoy, 1977] for a proof.

The next proposition will useful is showing that $\mathbb{Q}$ is not homeomorphic to any complete metric space in Section 7.1.

Proposition 4.5 (Preservation under homeomorphism)**.** *If two topological spaces X and Y are homeomorphic, and X is a Baire space, then so is Y.*

Proof. The proof is trivial since being an open subset and being a dense subset are both invariant under homeomorphisms, and since homeomorphisms are bijections, the image of the intersection is the intersection of the images. □

Proposition 4.6 (Closure Of Baire Subspace is Baire)**.** *Let Y be a dense subset of a topological space X such that it is Baire. Then X is also a Baire space.*

Proof. Suppose to the contrary X is not a Baire space. Then by Proposition 4.2 (c) there exists a nonempty open meager subset U of X. But then $Y \cap U$ is a nonempty open meager subset of Y by Proposition 3.4, which contradicts the fact that Y is a Baire space. □

For a topological space X, an arbitrary union of Baire subspaces of X need not be Baire since a singleton set is a Baire space. But we have the following two results if the Baire subspaces are open or if the union is of a finite family.

Proposition 4.7 (Arbitrary Union of Open Baire Spaces)**.** *In a topological space X, the union of any family of open Baire subspaces is a Baire space.*

Proof. Let $\mathcal{U}$ be a family of open Baire subspaces of X. Suppose V is an open meager subset of $\bigcup_{U\in\mathcal{U}} U$. Let $U \in \mathcal{U}$ be such that it intersects V. Then $U \cap V$ is open and meager in U. Therefore, by Proposition 4.2 (e), $\bigcup_{U\in\mathcal{U}} U$ is Baire since U is Baire. □

Proposition 4.8 (Finite Union of Baire Spaces)**.** *In a topological space X, the union of a finite family of Baire subspaces is a Baire space.*

Proof. It is sufficient to show the proposition for two Baire subspaces. Without loss of generality, we can let $X = Y \cup Z$, where Y and Z are Baire spaces, and we want to show that X is a Baire space.

$X \setminus \mathrm{cl}_X(Y)$ and $X \setminus \mathrm{cl}_X(Z)$ are open in Z and Y, respectively, and hence by Proposition 4.4, they both Baire spaces. By Proposition 4.7 then

$$B = (X \setminus \mathrm{cl}_X(Y)) \cup (X \setminus \mathrm{cl}_X(Z)) = X \setminus (\mathrm{cl}_X(Y) \cap \mathrm{cl}_X(Z))$$

is an open Baire subspace of X. By Proposition 4.6, $\mathrm{cl}_X(Y)$ is a Baire space, and since $X \setminus \mathrm{cl}_X(B)$ is its open subset, by Proposition 4.4, $X \setminus \mathrm{cl}_X(B)$ is a Baire space. But $B \cup (X \setminus \mathrm{cl}_X(B))$ is a Baire subspace by Proposition 4.7 and is dense in X, and therefore by Proposition 4.6, X is Baire. □

Products of Baire spaces may not be Baire space. The analysis is involved. See [Cohen, 1976] for details.

As is intuitively expected, a disjoint sum of Baire spaces is Baire.

Proposition 4.9 (Disjoint sums)**.** *Every disjoint topological sum of Baire spaces is a Baire space.*

Proof. For Baire spaces $\{X_\alpha\}_{\alpha\in A}$, let $X = \bigsqcup_{\alpha\in A} X_\alpha$ be their disjoint sum and let U be an open subset of X. By Proposition 4.4, $U \cap X_\alpha$ is a Baire space if the intersection is nonempty. Since $U = \bigcup_\alpha (U \cap X_\alpha)$ where the union is over all such α, by Proposition 4.7, U is a Baire space. X can now be written as a union of open Baire spaces and hence is Baire. □

The last property we consider is the relation of being Baire with the underlying topology. Suppose $(X, \mathcal{T})$ is a topological space. Then $(X, 2^X)$ is always a Baire space since it can be completely metrized using the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}, \quad x, y \in X.$$

Therefore, even if $(X, \mathcal{T})$ is a Baire space, $(X, \mathcal{S})$ may not be a Baire space if $\mathcal{S}$ is strictly coarser than $\mathcal{T}$.

5 BAIRE CATEGORY THEOREMS

In this section we prove the main results of this article, the Baire category theorems. They state that if X is a locally compact Hausdorff space or a complete pseudometric space, then X is a Baire space. These theorems will have many applications as we will see in Section 7. Proving both these results involves constructing a nested sequence of open sets and then showing that the intersection of their closures is nonempty. We start with proving the Baire category theorem for locally compact Hausdorff spaces, for which we will need a few results which we prove next.

Since locally compact spaces have different meanings in literature, let us define it for our use case.

Definition 5.1 (Locally Compact)**.** A topological space X is said to be *locally compact at* x if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said simply to be *locally compact*.

The first result we need is an equivalent characterization of locally compact Hausdorff spaces. This characterization will allow us to construct our nested sequence of open sets. But first we need a simple lemma.

Lemma 5.2 (A point outside a compact subspace in a Hausdorff space can be separated)**.** *If Y is a compact subspace of a Hausdorff space X and $x \in X \setminus Y$, then there exist disjoint open subsets U and V of X such that $x \in U$ and $Y \subseteq V$.*

Proof. For each $y \in Y$, there exists a pair of disjoint open sets (U_y, V_y) such that $x \in U_y$ and $y \in V_y$, because X is Hausdorff. By our construction, the collection $\{V_y\}_{y \in Y}$ is an open cover for the compact set Y, and therefore, there exists a finite subcover, say $\{V_i\}_{i=1}^n$. Define $V = \bigcup_{i=1}^n V_i$ and define $U = \bigcap_{y \in Y} U_y$, giving us the desired disjoint open sets. □

Proposition 5.3 (Equivalent characterization of local compactness in Hausdorff spaces)**.** *Let X be a Hausdorff space. Then X is locally compact if and only if given an open subset $U \subseteq X$ and a point $x \in U$, there is an open subset $V \subseteq X$ containing x such that $\mathrm{cl}_X(V)$ is compact and $\mathrm{cl}_X(V) \subseteq U$.*

Proof. One side is easy: Given a point $x \in U$, letting $U = X$ in the hypothesis, there is an open subset $V \subseteq X$ containing x such that $\mathrm{cl}_X(V)$ is compact, and therefore $\mathrm{cl}_X(V)$ serves as the compact set containing the neighborhood V of x.

For the other side, suppose X is a locally compact Hausdorff space, and let U be an open set in X with $x \in U$. Let X^* be the Alexandroff one-point compactification of X. then since U is open in X, it is also open in X^*, and thus $X^* \setminus U$ is closed in X^*. Since X^* is compact, $X^* \setminus U$ is a compact subspace of X^*. Since X^* is Hausdorff, we can use Lemma 5.2 to find disjoint open subsets V and W of X^* such that $x \in V$ and $X^* \setminus U \subseteq W$. The closure $\mathrm{cl}_{X^*}(V)$ is compact and satisfies $\mathrm{cl}_{X^*}(V) \cap (X^* \setminus U) = \varnothing$. Therefore, $\mathrm{cl}_{X^*}(V) = \mathrm{cl}_X(V)$ is compact and satisfies $\mathrm{cl}_X(V) \subseteq U$, as desired. □

We next prove a proposition that allows us to make claims about the nonemptyness of intersection of a family of closed sets satisfying the finite intersection property. Recall that a collection $\mathcal{C}$ of subsets of X is said to have the *finite intersection property* if every finite subcollection of $\mathcal{C}$ has nonempty intersection.

Proposition 5.4 (Equivalent characterization of compactness)**.** *A topological space X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ is nonempty.*

Proof. X is compact means that given any collection $\mathcal{U}$ of open sets, if no finite subcollection covers X, then $\mathcal{U}$ cannot cover X. Define the collection $\mathcal{C} = \{X \setminus U \ : \ U \in \mathcal{U}\}$ of closed sets in X, and note that we could have constructed $\mathcal{U}$ given $\mathcal{C}$ similarly. It is easy to see that the collection $\mathcal{U}$ does not cover X if and only if $\bigcap_{C \in \mathcal{C}} C$ is nonempty, and similarly a finite subcollection $\{U_1, \ldots, U_n\}$ of $\mathcal{U}$ does not cover X if and only if the intersection $\bigcap_{i=1}^n C_i$ of the corresponding closed sets from $\mathcal{C}$ is nonempty.

Therefore, the statement given any collection $\mathcal{U}$ of open sets, if no finite subcollection covers X, then $\mathcal{U}$ cannot cover X is equivalent to the statement given any collection $\mathcal{C}$ of closed sets, if every finite intersection of elements of $\mathcal{C}$ is nonempty, then the intersection $\bigcap_{C \in \mathcal{C}} C$ is nonempty. □

We have now done most of the legwork to establish the Baire category theorem for locally compact Hausdorff spaces.

Theorem 5.5 (Baire category theorem for locally compact Hausdorff space)**.** *Every locally compact Hausdorff space X is a Baire space.*

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be a collection of dense open sets in X. We want to show that their intersection is also dense in X, i.e., we want to show that for any nonempty open subset $V \subseteq X$, the intersection $V \cap \left(\bigcap_{n\in\mathbb{N}} U_n\right)$ is nonempty. To that end, fix an arbitrary nonempty open subset $V \subseteq X$. Since U_1 is dense in X, there exists a point $x_1 \in V \cap U_1$. Proposition 5.3 implies that there exists an open subset $V_1 \subseteq X$ containing x_1 such that $\mathrm{cl}_X(V_1)$ is compact and $\mathrm{cl}_X(V_1) \subseteq V \cap U_1$. Similarly, for $n > 1$, we use Proposition 5.3 to get a nonempty open subset $V_n \subseteq X$ such that $\mathrm{cl}_X(V_n)$ is compact and $\mathrm{cl}_X(V_n) \subseteq V_{n-1} \cap U_n$.

The sequence $\mathrm{cl}_X(V_1) \supseteq \mathrm{cl}_X(V_2) \supseteq \cdots$ is a collection of closed sets in the compact space $\mathrm{cl}_X(V_1)$ satisfying the finite intersection property, and therefore by Proposition 5.4, the intersection $\bigcap_{n\in\mathbb{N}} \mathrm{cl}_X(V_n)$ is nonempty. But note that by our construction $\bigcap_{n\in\mathbb{N}} \mathrm{cl}_X(V_n) \subseteq V \cap \left(\bigcap_{n\in\mathbb{N}} U_n\right)$, showing that $V \cap \left(\bigcap_{n\in\mathbb{N}} U_n\right)$ is nonempty. □

The proof of Baire category theorem for complete pseudometric space proceeds in a very similar fashion, except here we use completeness instead of compactness to show that the intersection of the closures of the sequence of open sets constructed is nonempty.

Theorem 5.6 (Baire category theorem for complete pseudometric space)**.** *Every complete metric space (X, d) is a Baire space.*

Proof. (X, d) is equipped with the pseudometric topology generated by the open balls $B_r(p) = \{x \in X \,:\, d(x, p) < r\}$ for $p \in X$ and $r > 0$, which form a basis for the topology. For any subset A of X, denote by $d(A) = \sup\{d(s,t) \,:\, s, t \in A\}$ the diameter of A, and define $d^*(A) := \min\{1, d(A)\}$. Observe that if $A \subseteq B \subseteq X$ then $d^*(A) \leq d^*(B)$.

Let $\{U_n\}_{n\in\mathbb{N}}$ be a collection of dense open sets in X. We want to show that their intersection is also dense in X, i.e., we want to show that for any nonempty open subset $V \subseteq X$, the intersection $V \cap \left(\bigcap_{n\in\mathbb{N}} U_n\right)$ is nonempty.

To that end, fix an arbitrary nonempty open subset $V \subseteq X$. Since U_1 is dense in X, there exists a point $x_1 \in V \cap U_1$. By the definition of a basis, we can choose $r \in (0, 1)$ such that $B_r(x_1) \subseteq V \cap U_1$. If we define $r' := d(B_r(x_1))$, then by triangle inequality $r'/2 \leq r < 1$. Thus if we define $V_1 = B_{r'/4}(x_1)$ we have $\mathrm{cl}_X(V_1) \subseteq \{x \in X \,:\, d(x, x_1) \leq r'/4\} \subseteq B_r(x_1) \subseteq V \cap U_1$, and by triangle inequality $d(V_1) \leq r'/2 = d^*(B_r(p))/2 \leq d^*(V \cap U_1)/2 \leq d(V \cap U_1)/2$. Similarly, for $n > 1$, we construct a nonempty open subset $V_n \subseteq X$ such that $\mathrm{cl}_X(V_n) \subseteq V_{n-1} \cap U_n$ and $d(V_n) \leq d^*(V_{n-1})/2 \leq d(V_{n-1})/2$.

We claim that $\bigcap_{n\in\mathbb{N}} \mathrm{cl}_X(V_n)$ is nonempty. Since $d^*(V_1) \leq 1$ and $d(V_{n+1}) \leq d^*(V_n)/2$, we have $d(V_n) \leq 2^{-n+1}$ for each $n \in \mathbb{N}$. Each of these sets is nonempty and therefore we can choose $x_n \in V_n$ for each $n \in \mathbb{N}$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ becomes a Cauchy sequence and therefore since X is complete there exists $x \in X$ such that it is the limit of this sequence. By our construction, if $m \leq n$, then $x_n \in V_m \subseteq \mathrm{cl}_X(V_m)$. Thus, $x \in \mathrm{cl}_X(V_n)$ for each $n \in \mathbb{N}$. This shows that $x \in \bigcap_n \mathrm{cl}_X(V_n)$.

But note that by our construction $\bigcap_{n\in\mathbb{N}} \mathrm{cl}_X(V_n) \subseteq V \cap \left(\bigcap_{n\in\mathbb{N}} U_n\right)$, showing that $V \cap \left(\bigcap_{n\in\mathbb{N}} U_n\right)$ is nonempty. □

As a trivial (but useful!) corollary a complete metric space is a Baire space.

Locally compact Hausdorff spaces and completely metrizable spaces are of very different nature, and therefore it is surprising that they both are Baire spaces and therefore have many common properties. As an example of a locally compact Hausdorff that isn't a complete metric space, consider $[0, 1]^J$ for uncountable set J or $(0, 1)$ with the subspace topology with respect to $\mathbb{R}$. In fact, $[0, 1]^J$ is a compact Hausdorff space that isn't even metrizable. On the other hand, $\mathbb{N}^{\mathbb{N}}$ (which is homeomorphic to the space of irrationals) and infinite-dimensional Hilbert spaces are examples of complete metric spaces that aren't locally compact Hausdorff. Finally, the Sorgenfrey plane and the Niemytzki/Moore plane are Baire spaces which are neither complete pseudometric spaces nor a locally compact Hausdorff spaces.

6 TOPOLOGICAL GAMES

In a topological game players choose some objects related to the topological structure of a space, such as points, closed subsets, open covers, etc., and condition on a play to be winning for a player could also involve topological notions such as closure, a convergence, etc. Remarkably, it turns out that topological games can be used to define notions such as Baire property, Baire spaces, completeness properties, convergence properties, separation properties, continuous images, Suslin sets, etc. Viewing these topological notions through the lens of topological games often provides more insight and is even sometimes a more natural perspective.

In this section we will discuss two 2-player games, Banach-Mazur game and Choquet game. These games are closely tied to meager sets and Baire spaces.

In the period 1930s and 1940s mathematicians from the Lwów school in Poland, which included Banach, Ulam, Kuratowski, Steinhaus, Mazur, Kac, Alexandroff, von Neumann, Orlicz, Sobolev, Borsuk, Zygmund, Schauder, Eilenberg among others, collaboratively discussed problems, mainly in topology and functional analysis. In 1935, Banach started a famous notebook, called the Scottish Book, where these mathematicians proposed mathematical conjectures. In the same year Mazur proposed a topological game related to the Baire category theorem. A solution was proposed by Banach and therefore the game is now known as Banach-Mazur game [cite Telgarsky and Ulam's book from Oxtoby]. It was the first infinite positional game of perfect information to be studied.

6.1 Banach-Mazur Game

Let us state and prove the Banach-Mazur game as it proposed originally by Mazur. Instead of general topological spaces X we will be limiting ourselves to sets on the real line equipped with the usual Euclidean topology. We use the phrase closed interval to refer to intervals of the form $[a, b]$ for $a < b$.

Definition 6.1 (Banach-Mazur Game)**.** Player $\mathcal{P}_A$ is dealt an arbitrary subset A of a closed interval I_0. The complement $B = I_0 \setminus A$ is dealt to the player $\mathcal{P}_B$. The game, denoted by $\mathrm{BM}(A, B)$, is played as follows: $\mathcal{P}_A$ chooses any closed interval $I_1 \subseteq I_0$; then $\mathcal{P}_B$ chooses any closed interval $I_2 \subseteq I_1$; and so on, alternatively. Together the players determine a nested sequence $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ of closed intervals, $\mathcal{P}_A$ choosing those with odd index and $\mathcal{P}_B$ choosing those with even index. $\mathcal{P}_A$ wins if and only if $A \cap \left(\bigcap_{n\in\mathbb{N}} I_n\right) \neq \varnothing$; otherwise, $\mathcal{P}_B$ wins.

The following theorem due to Mazur states that if A is meager then there exists a strategy which $\mathcal{P}_B$ can use to win the game no matter what $\mathcal{P}_A$ plays.

Theorem 6.2. *If I_0 is any closed interval and $A \cup B$ a partition of I_0, then there exists a winning strategy for $\mathcal{P}_B$ for the game* $\mathrm{BM}(A, B)$ *if A is meager in I_0.*

Proof. Write $A = \bigcup_{n\in\mathbb{N}} A_n$ as a countable union of nowhere dense sets in I_0. Then at each step $n \in \mathbb{N}$, $\mathcal{P}_B$ can choose a closed interval I_{2n} such that $I_{2n} \subseteq I_{2n-1} \setminus A_n$. The $A \cap \left(\bigcap_{n\in\mathbb{N}} I_{2n}\right) = \varnothing$, and therefore no matter how $\mathcal{P}_A$ plays, $\mathcal{P}_B$ will win. □

Mazur conjectured that *only* when A is meager in I_0 does there exist a strategy which $\mathcal{P}_B$ can use to win. Banach proved this result. Before we give the proof, let us formalize what we mean by a "strategy".

Definition 6.3 (Strategy)**.** A *strategy* for either player is a sequence of closed-interval-valued functions each of which specify the choice of the set by the player in the corresponding move while satisfying the rules of the game. More specifically, for $\mathcal{P}_B$ denote the strategy by the sequence $\{f_n\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$,

$$I_{2n} = f_n(I_0, I_1, \ldots, I_{2n-1}) \subseteq I_{2n-1}. \tag{1}$$

Then this is a *winning strategy* for $\mathcal{P}_B$ if and only if $\bigcap_{n\in\mathbb{N}} I_n \subseteq B$ for any valid choice of $\{I_{2n-1}\}_{n\in\mathbb{N}}$ by $\mathcal{P}_A$.

Theorem 6.4. *If I_0 is any closed interval and $A \cup B$ a partition of I_0, then there exists a winning strategy for $\mathcal{P}_B$ for the game* $\mathrm{BM}(A, B)$ *if and only if A is meager in I_0.*

Proof. We have already proved the if part in Theorem 6.2. So assume that $\{f_n\}_{n\in\mathbb{N}}$ is a winning strategy for $\mathcal{P}_B$. We want to show that A is meager, which we do by showing that A is a subset of a countable union $\bigcup_{n\in\mathbb{N}}(I_0\setminus G_n)$ of nowhere dense sets. This we will do by constructing open sets $\{G_n\}_{n\in\mathbb{N}}$ such that each G_n is dense in I_0. The construction proceeds by induction to first construct clever families of sets $\{J_{i_1,\ldots,i_n} : n, i_1, \ldots, i_n \in \mathbb{N}\}$ and $\{K_{i_1,\ldots,i_n} : n, i_1, \ldots, i_n \in \mathbb{N}\}$ which are used to define $\{G_n\}_{n\in\mathbb{N}}$ and also a particular sequence $\{I_n\}_{n\in\mathbb{N}}$ for which, since $\{f_n\}_{n\in\mathbb{N}}$ is a winning strategy, we will use the fact that $\bigcap_{n\in\mathbb{N}} I_n \subseteq B$. To that end let us start by constructing the families $\{J_{i_1,\ldots,i_n} : n, i_1, \ldots, i_n \in \mathbb{N}\}$ and $\{K_{i_1,\ldots,i_n} : n, i_1, \ldots, i_n \in \mathbb{N}\}$.

The collection $\mathcal{S}$ of all closed intervals that have rational endpoints and are contained in $\mathrm{int}_{\mathbb{R}}(I_0)$ is countable and therefore $\mathcal{S}$ can be thought of as a sequence. We define two sequences $\{J_n\}_{n\in\mathbb{N}}$ and $\{K_n = f_1(I_0, J_n)\}_{n\in\mathbb{N}}$ of closed intervals as follows: J_1 is the first element of $\mathcal{S}$; for $n > 1$, define J_n to be the first term of $\mathcal{S}$ contained in $I_0 \setminus \left(\bigcup_{i=1}^{n-1} K_i\right)$. We then have

(i) for each $n \in \mathbb{N}$, $J_n \subseteq \mathrm{int}_{\mathbb{R}}(I_0)$,

(ii) the sets $\{K_n\}_{n\in\mathbb{N}}$ are pairwise disjoint, and

(iii) $\bigcup_{n\in\mathbb{N}} \mathrm{int}_{\mathbb{R}}(K_n)$ is dense in I_0.

The first and the second property are obvious from the construction and the fact that $K_n \subseteq f_1(I_0, J_n)$. The third property follows from the fact that rationals are dense in $\mathbb{R}$, and therefore for every open subset of I_0 we can find a closed interval with rational endpoints lying inside this open set. Note that the elements of the sequence $\{J_n\}_{n\in\mathbb{N}}$ are contained in $\mathrm{int}_{\mathbb{R}}(I_0)$.

Similarly, for each $m \in \mathbb{N}$, let $\{J_{m,n}\}_{n\in\mathbb{N}}$ be a sequence of closed intervals contained in $\mathrm{int}_{\mathbb{R}}(K_m)$ such that the sets $\{K_{m,n} = f_2(I_0, J_m, K_m, J_{m,n})\}_{n\in\mathbb{N}}$ are pairwise disjoint, and $\bigcup_{n\in\mathbb{N}} \mathrm{int}_{\mathbb{R}}(K_{m,n})$ is dense in K_m. Now recall Proposition 2.6, which implies that $\bigcup_{m,n\in\mathbb{N}} \mathrm{int}_{\mathbb{R}}(K_{m,n})$ is dense in I_0.

Proceeding inductively, we define two families of closed intervals

$$\{J_{i_1,\ldots,i_n} : n, i_1, \ldots, i_n \in \mathbb{N}\}\text{, and}$$
$$\{K_{i_1,\ldots,i_n} = f_n(I_0, J_{i_1}, K_{i_1}, J_{i_1,i_2}, K_{i_1,i_2}, \ldots, J_{i_1,\ldots,i_n}) : n, i_1, \ldots, i_n \in \mathbb{N}\}$$

satisfying the following properties

(i) $J_{i_1,\ldots,i_n} \subseteq \mathrm{int}_{\mathbb{R}}(K_{i_1,\ldots,i_n})$,

(ii) for each $n \in \mathbb{N}$, the sets $\{K_{i_1,\ldots,i_n} : i_1, \ldots, i_n \in \mathbb{N}\}$ are pairwise disjoint, and

(iii) for each $n \in \mathbb{N}$, $\bigcup_{i_1,\ldots,i_n\in\mathbb{N}} \mathrm{int}_{\mathbb{R}}(K_{i_1,\ldots,i_n})$ is dense in I_0.

Having defined these families, note that if $\{i_n\}_{n\in\mathbb{N}}$ is any arbitrary sequence of positive integers, then the sequence $\{I_n\}_{n\in\mathbb{N}}$ such that

$$I_{2n-1} = J_{i_1,\ldots,i_n} \text{ and } I_{2n} = K_{i_1,\ldots,i_n}, \quad n \in \mathbb{N} \tag{2}$$

is a valid play of the game consistent with the strategy $\{f_n\}_{n\in\mathbb{N}}$ of $\mathcal{P}_B$. By hypothesis, $\{f_n\}_{n\in\mathbb{N}}$ is a winning strategy, and therefore $\bigcap_{n\in\mathbb{N}} I_n \subseteq B$.

Define the sequence $\{G_n\}_{n\in\mathbb{N}}$ of open sets by

$$G_n = \bigcup_{i_1,\ldots,i_n\in\mathbb{N}} \mathrm{int}_{\mathbb{R}}(K_{i_1,\ldots,i_n}),$$

and let $E = \bigcap_{n\in\mathbb{N}} G_n$ be their intersection. Then $E \subseteq B$ because if $x \in E$, then x is in every G_n, which with property (ii) gives a unique sequence $\{i_n\}_{n\in\mathbb{N}}$ such that $x \in K_{i_1,\ldots,i_n}$ for every $n \in \mathbb{N}$, which if used to define $\{I_n\}_{n\in\mathbb{N}}$ like in (2) and noting $K_{i_1,\ldots,i_n} \subseteq J_{i_1,\ldots,i_n}$ gives $x \in \bigcap_{n\in\mathbb{N}} I_n \subseteq B$. But now note that

$$A = I_0 \setminus B \subseteq I_0 \setminus E = \bigcup_{n\in\mathbb{N}} (I_0 \setminus G_n).$$

By the definition of G_n and property (iii), $I_0 \setminus G_n$ is nowhere dense in I_0 for each $n \in \mathbb{N}$. This finishes the proof. □

This theorem gives new insight into the sense in which a meager set is "small" – even the advantage of being the first player is not enough. When can the player $\mathcal{P}_A$ be sure to win? This is answered by the next theorem, and is a simple consequence of our previous theorem.

Theorem 6.5. *If I_0 is any closed interval and $A \cup B$ a partition of I_0, then there exists a winning strategy for $\mathcal{P}_A$ for the game* $\mathrm{BM}(A, B)$ *if and only if B is meager in some closed interval $I_1 \subseteq I_0$.*

Proof. If such a closed interval I_1 exists, then $\mathcal{P}_A$ can start by choosing I_1 and then by following the strategy of $\mathcal{P}_B$ in Theorem 6.2, $\mathcal{P}_A$ can ensure than $B \cap \left(\bigcap_{n\in\mathbb{N}} I_n\right) = \varnothing$. The intersection $\bigcap_{n\in\mathbb{N}} I_n$ is nonempty (see the proof of Theorem 5.6 where a generalization of this well-known fact was proved), and therefore this is a winning strategy for $\mathcal{P}_A$.

On the other hand, if $\mathcal{P}_A$ has a winning strategy, then suppose he chooses I_1 as his first move. After this move, the game $\mathrm{BM}(A, B)$ becomes the game $\mathrm{BM}(I_1 \cap B, I_1 \cap A)$ and player $\mathcal{P}_A$ becomes player $\mathcal{P}_{I_1\cap A}$. Then using Theorem 6.4 $\mathcal{P}_{I_1\cap A}$ has a winning strategy only if $I_1 \cap B$ is meager in I_1. □

Is it possible that neither player has a winning strategy? Yes, for example, if $A = I_0 \cap B$, where B is the Bernstein set defined to be a subset of $\mathbb{R}$ such that both B and $\mathbb{R} \setminus B$ intersect every uncountable closed set. For a proof of this result, see Chapter-6 in [Oxtoby, 1971].

6.1.1 Generalization

In [Oxtoby, 1957], Oxtoby proved a generalization of the above game. The game is still called the Banach-Mazur game. We will not be proving this result, but will state it for completeness.

Definition 6.6 (Banach-Mazur Game)**.** Let X be any topological space, and let $\mathcal{G}$ be a specified class of subsets of X such that

(i) each element of $\mathcal{G}$ has a nonempty interior in X, and

(ii) every nonempty open subset of X contains an element of $\mathcal{G}$.

Let $X = A \cup B$ be an arbitrary partition of X. The game $\mathrm{BM}(A, B)$ is played by two players $\mathcal{P}_A$ and $\mathcal{P}_B$ as follows: $\mathcal{P}_A$ chooses any $G_1 \in \mathcal{G}$; then $\mathcal{P}_B$ chooses any $G_2 \in \mathcal{G}$ such that $G_2 \subseteq G_1$; and so on alternatively. Together the players determine a nested sequence $G_1 \subseteq G_2 \subseteq \cdots$ of elements from $\mathcal{G}$, with $\mathcal{P}_A$ choosing those with odd index and $\mathcal{P}_B$ choosing those with even index.

$\mathcal{P}_A$ wins if and only if $A \cap \left(\bigcap_{n\in\mathbb{N}} G_n\right) \neq \varnothing$; otherwise $\mathcal{P}_B$ wins.

Theorem 6.7 (Oxtoby (1957))**.** *(i) Using the notion above, there exists a winning strategy for $\mathcal{P}_B$ for the game* $\mathrm{BM}(A, B)$ *if and only if A is meager in X.*

(ii) If we assume that X is a complete metric space, there exists a winning strategy for $\mathcal{P}_A$ for the game $\mathrm{BM}(A, B)$ *if and only if B is meager in some nonempty open subset of X.*

Proof sketch. The proof is very similar to the proof of the Theorems 6.4 and 6.5. For details see [Oxtoby, 1957]. □

6.2 Choquet Game

Choquet game proposed by Choquet in his book [Choquet, 1969] is closely related to the Banach-Mazur game.

Definition 6.8 (Choquet Game)**.** Let X be a nonempty topological space. The *Choquet game* C_X of X is a game with two players $\mathcal{P}_X$ and $\mathcal{P}_\varnothing$ taking turns in choosing nonempty open subsets of X as follows: Player $\mathcal{P}_X$ chooses G_1, then player $\mathcal{P}_\varnothing$ chooses $G_2 \subseteq G_1$; and so on alternatively. Together the players determine a nested sequence $G_1 \subseteq G_2 \subseteq \cdots$ of nonempty open subsets of X, with $\mathcal{P}_X$ choosing those with odd index and $\mathcal{P}_\varnothing$ choosing those with even index.

$\mathcal{P}_\varnothing$ wins this game if and only if $\bigcap_{n\in\mathbb{N}} G_n \neq \varnothing$; otherwise $\mathcal{P}_X$ wins.

Note the similarity between this game and the Banach-Mazur game specified by taking $A = X$ and $\mathcal{G}$ to be the collection of all nonempty open subsets of X in Definition 6.6. In the Banach-Mazur game we say $\mathcal{P}_X$ wins $\mathrm{BM}(X, \varnothing)$ if and only if $\bigcap_{n\in\mathbb{N}} G_n \neq \varnothing$, while in the Choquet game we say $\mathcal{P}_\varnothing$ wins C_X if and only if $\bigcap_{n\in\mathbb{N}} G_n \neq \varnothing$. This swap changes a lot though as the following theorem demonstrates.

Theorem 6.9. *A nonempty topological space X is a Baire space if and only if player $\mathcal{P}_X$ has no winning strategy in the Choquet game C_X.*

Having shown the relation between the Choquet and Banach-Mazur games, let us change the notation a little for convenience. Let us denote the players $\mathcal{P}_X$ and $\mathcal{P}_\varnothing$ by $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively for players 1 and 2. The choices of $\mathcal{P}_1$ which is the collection $\{G_{2n-1}\}_{n\in\mathbb{N}}$ is denoted by $\{U_n\}_{n\in\mathbb{N}}$ and the choices of $\mathcal{P}_2$ which is the collection $\{G_{2n}\}_{n\in\mathbb{N}}$ is denoted by $\{V_n\}_{n\in\mathbb{N}}$. The strategy for $\mathcal{P}_1$ is the sequence of nonempty-open-subset-valued functions $\{f_n\}_{n\in\mathbb{N}}$ such that f_1 gives the first choice U_1 of $\mathcal{P}_1$ and the remaining functions satisfy

$$U_n = f_n(U_1, V_1, \ldots, V_{n-1}) \subseteq V_{n-1}.$$

Then $\{f_n\}_{n\in\mathbb{N}}$ is a winning strategy for $\mathcal{P}_1$ if and only if $\bigcap_{n\in\mathbb{N}} U_n = \varnothing$.

Proof sketch. We will prove the contrapositives in each direction. For the if side, assume that X is not a Baire space. We will show that $\mathcal{P}_1$ has a winning strategy by constructing a sequence of nonempty open sets $\{U_n\}_{n\in\mathbb{N}}$ such that their intersection is empty.

Since X isn't a Baire space Proposition 4.2 (b) implies that there exists nonempty open set U_1 in X and a sequence $\{O_n\}_{n\in\mathbb{N}}$ of dense open sets in X such that $U_1 \cap \left(\bigcap_{n\in\mathbb{N}} O_n\right) = \varnothing$. This nonempty open set U_1 becomes the first move for $\mathcal{P}_1$. Suppose $\mathcal{P}_2$ now plays $V_1 \subseteq U_1$ for any nonempty open set V_1. Note that $V_1 \cap O_1 \neq \varnothing$ since O_1 is dense in X. Let $\mathcal{P}_1$ play $U_2 = V_1 \cap O_1 \subseteq V_1$. In the n^{th} move, $\mathcal{P}_1$ plays $U_n = V_{n-1} \cap O_n \subseteq V_{n-1}$, which is nonempty because O_n is dense in X. But then we have

$$\bigcap_{n\in\mathbb{N}} U_n \subseteq U_1 \cap \left(\bigcap_{n\in\mathbb{N}} O_n\right) = \varnothing.$$

For the other side, suppose $\mathcal{P}_1$ has a winning strategy $\{f_n\}_{n\in\mathbb{N}}$, and U_1 is his first move. We will show that U_1 is not Baire, which coupled with Proposition 4.4 will show that X is not Baire. Then proceeding as we do in proofs of Theorems 6.4 and 6.7 we construct a clever family of open sets through which we construct a sequence of open dense sets $\{G_n\}_{n\in\mathbb{N}}$ in U_1. We then show that their intersection is empty, thereby showing that U_1 is not Baire. □

7 APPLICATIONS

This section gives a few interesting applications of Baire category theorems. One of the most common applications of Baire category theorems is in proving the three big guns of functional analysis, namely Open Mapping Theorem, Closed Graph Theorem, Uniform Boundedness Principle. But these proofs can can be found very easily in any functional analysis text (see [Megginson, 1998] for example), so we will skip them.

Usually these applications follow a similar recipe: Suppose that S is a desired property for the elements of some nonempty complete pseudo-metric space or some locally compact Hausdorff space X. One way to show that S is nonempty is to show that $X \setminus S$ is meager in X. This is therefore usually an overkill, and why results like these are often surprising.

7.1 $\mathbb{Q}$ Is Not Homeomorphic To Any Complete Metric Space

It is well-known that $\mathbb{Q}$, with the subspace topology with respect to $\mathbb{R}$ with its usual Euclidean topology, is not a complete metric space. But completeness is not a topological property as can be seen with the example of $\mathbb{R}$ and $(0, 1)$, both equipped with the usual topology, being homeomorphic under the homeomorphism

$$(0, 1) \ni x \mapsto \tan\left(-\frac{\pi}{2} + \pi x\right) \in \mathbb{R},$$

but $\mathbb{R}$ is complete while $(0,1)$ is not. It is a simple consequence of Baire category theorem for complete metric spaces that $\mathbb{Q}$ is not homeomorphic to any complete metric spaces.

Theorem 7.1. *$\mathbb{Q}$ is not homeomorphic to any complete metric spaces.*

Proof. Let's first show that $\mathbb{Q}$ is not a Baire space. Indeed, singletons in $\mathbb{Q}$ are closed and have empty interior, and therefore $\mathbb{Q}$, which can be written as a countable union of all its singletons, cannot be Baire. Now by Proposition 4.5 and by Baire category theorem for complete metric spaces, $\mathbb{Q}$ cannot be homeomorphic to any complete metric spaces. □

7.2 Infinite-Dimensional Banach Spaces

We start with a very simple application of Baire's category theorem.

Theorem 7.2. *If X is an infinite dimensional Banach space, then every Hamel basis of X is uncountable.*

Proof. Suppose to the contrary that $\{v_n\}_{n\in\mathbb{N}}$ is a countable Hamel basis for an infinite dimensional Banach space X. For each $n \in \mathbb{N}$, define $F_n = \langle v_1, \dots, v_n\rangle$ to be the span of the first n vectors. Then by the definition of a Hamel basis, $X = \bigcup_{n\in\mathbb{N}} F_n$. Recall at this point that X is a complete metric space, and therefore a Baire space.

Now note that each F_n, being a finite-dimensional normed space, is complete. A complete subspace of a complete metric space is closed, and therefore F_n is closed in X.

Next, we prove that each F_n is nowhere dense. Since it is closed, we just need to show that F_n has empty interior in X. Suppose to the contrary that is not the case, and there exists an open ball $B_r(x_0) = \{y \in X : \|y - x\| < r\}$ centered at some point $x_0 \in X$ for some $r > 0$ satisfying $B_r(x_0) \subseteq F_n$. We now note that every point in X can be rescaled and translated so that the transformation belongs to the ball $B_r(x_0)$: if $x \in X$, then $z = x_0 + \frac{r}{2\|x\|}x \in B_r(x_0)$. Since F_n is a subspace, this implies that $x = \frac{2\|x\|}{r}(z - x_0) \in F_n$, implying $X \subseteq F_n$, a contradiction.

These two facts together with the fact that X is a Baire space imply that X has empty interior, which is absurd.

□

A similar reasoning can be used to prove that the linear space of all polynomials in one variable P is not a Banach space in any norm. This is because P can be written as $P = \bigcup_{n\in\mathbb{N}} P_n$, where P_n is the subspace of P containing polynomials of degree at most n. Each P_n is closed and nowhere dense in P, and therefore P cannot be a Baire space, which, in particular, means that it cannot be a Banach space.

7.3 A Generic Element Of $C[0,1]$ Is Nowhere Differentiable

Let us first make precise the meaning of "generic".

Definition 7.3. Let X be a topological space and $S \subseteq X$. If $X \setminus S$ is meager, we say that S *holds generically*, or that a *generic element of* X is in S. (Sometimes the word *typical* is used instead of generic.)

Definition 7.4. Let $C[0,1]$ denote the collection of all real-valued continuous functions on $[0,1]$. We endow this collection with the *sup norm*,

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|, \quad f \in C[0,1].$$

This makes $C[0,1]$ into a Banach space (see [Megginson, 1998] for a proof).

We now prove that a generic element of $C[0,1]$ is nowhere differentiable. In fact, we will prove something stronger.

Theorem 7.5. *Let D_+ be the collection of all members f of $C[0,1]$ for which there is a point $x_f \in [0,1)$ at which f has a finite right-hand derivative. Then D_+ is meager in $C[0,1]$.*

Proof. We first note that $C[0,1]$ being a complete metric space is a Baire space by Theorem 5.6, and therefore the intersection of countably many dense open sets $\{U_n\}_{n\in\mathbb{N}}$ in $C[0,1]$ is dense in $C[0,1]$. This then implies that $C[0,1] \setminus \bigcap_{n\in\mathbb{N}} U_n$ is meager in $C[0,1]$. Our strategy will be to define this sequence $\{U_n\}_{n\in\mathbb{N}}$ in such a way that $D_+ \subseteq C[0,1] \setminus \bigcap_{n\in\mathbb{N}} U_n$, thereby showing that D_+ is meager in $C[0,1]$.

To this end, let

$$U_n = \left\{ f \in C[0,1] \ : \ \sup \left\{ \left| \frac{f(y)-f(x)}{y-x} \right| \ : \ y \in \left(x, x+\frac{1}{n}\right) \right\} > n, \ \forall\, x \in \left[0, 1-\frac{1}{n}\right] \right\},$$

and note that a function $f \in C[0,1]$ with a finite right-hand derivative at some point in $[0,1)$ cannot lie in every U_n. Therefore, $D_+ \subseteq C[0,1] \setminus \bigcap_{n\in\mathbb{N}} U_n$. What remains to show that each U_n is open and dense in $C[0,1]$. Fix an arbitrary $n_0 \in \mathbb{N}$.

U_{n_0} **is open in** $C[0,1]$**:** We show this by showing that $C[0,1] \setminus U_{n_0}$ is closed in $C[0,1]$. Suppose $\{f_m\}_{m\in\mathbb{N}}$ is a sequence in $C[0,1] \setminus U_{n_0}$ that converges to $f_0 \in C[0,1]$. Since $C[0,1]$ is a metric space, to show $C[0,1] \setminus U_{n_0}$ is closed in $C[0,1]$, it is sufficient to show that $f_0 \in C[0,1] \setminus U_{n_0}$. By the definition of U_{n_0}, for each $m \in \mathbb{N}$, there exists $x_m \in [0, 1-1/n_0]$ such that

$$\left| \frac{f_m(y) - f_m(x_m)}{y - x_m} \right| \le n_0, \quad \forall\, y \in \left(x_m, x_m + \frac{1}{n_0} \right).$$

The sequence $\{x_m\}_{m\in\mathbb{N}}$ lies in the compact set $[0, 1-1/n_0]$, and therefore there exists a convergent subsequence, say $\{x_{m_k}\}_{k\in\mathbb{N}}$ converging to $x_0 \in [0, 1-1/n_0]$. If $y \in (x_0, x_0 + 1/n_0)$, then for k large enough $y \in (x_{m_k}, x_{m_k} + 1/n_0)$. Since the functions $\{f_m\}_{m\in\mathbb{N}}$ are continuous, we have

$$\left| \frac{f_{m_k}(y) - f_{m_k}(x_{m_k})}{y - x_{m_k}} \right| \xrightarrow{k\to\infty} \left| \frac{f_0(y) - f_0(x_0)}{y - x_0} \right|.$$

Therefore, $\left| \frac{f_0(y)-f_0(x_0)}{y-x_0} \right| \ge n_0$, and we have $f_0 \in C[0,1] \setminus U_{n_0}$.

U_{n_0} **is dense in** $C[0,1]$**:** Recall the Stone-Weierstrass theorem [Folland, 1999] which implies that polynomials on $[0,1]$ (with the subspace topology), denoted $P[0,1]$, are dense in $C[0,1]$. Therefore, in light of Proposition 2.6, it is sufficient to show that U_{n_0} is dense in $P[0,1]$. Given $\epsilon > 0$ and $M > 0$, Figure 1 shows how to construct a function in $C[0,1]$ that has norm ϵ and absolute value of right-hand derivative greater than M at each point of $[0,1)$.

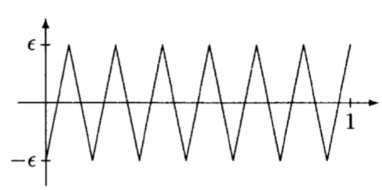

Figure 1: A member of $C[0,1]$ with norm ϵ and large right-hand derivative. (Image take from [Megginson, 1998])

Since each polynomial $p \in P[0,1]$ has a bounded right-hand derivative on $[0,1)$, it follows that a function of the form in Figure 1 can be added to p to obtain a member $u \in U_{n_0}$ such that $\|u-p\|_\infty \le \epsilon$. This shows that U_{n_0} is dense in $P[0,1]$.

∎

7.4 A Special Infinitely Differentiable Function

We end our applications with a very curious result.

Theorem 7.6. *If $f\colon \mathbb{R} \to \mathbb{R}$ is infinitely differentiable, and for every $x \in \mathbb{R}$ there is a nonnegative integer n such that $f^{(n)}(x) = 0$, then f is a polynomial.*

Proof. We will prove by contradiction. So suppose f is not a polynomial. Define the set

$$X = \left\{ x \in \mathbb{R} \ :\ \text{for every interval } (a,b) \text{ containing } x, \text{ the restriction } f\restriction_{(a,b)} \text{ is not a polynomial} \right\},$$

and note that since f is not a polynomial, X is nonempty. It is easy to see that X is closed with all of its points being limit points. Define the sequence of sets $\{S_n\}_{n\in\mathbb{N}}$ by

$$S_n = \left\{ x \ :\ f^{(n)}(x) = 0 \right\},$$

and note that for each $n \in \mathbb{N}$, by the continuity of $f^{(n)}$, S_n is closed. Using the property of f in the hypothesis, we can write

$$X = \bigcup_{n\in\mathbb{N}} (X \cap S_n)\,.$$

Now note that X being closed is also complete, and therefore by the Baire category theorem is a Baire space. Since each $X \cap S_n$ is closed in X, this implies that there exists $N \in \mathbb{N}$ such that $X \cap S_N$ is not nowhere dense in X. Equivalently, this means there exists a nonempty open subset of X contained in $X \cap S_N$. So there exists an interval (a, b) such that

(i) $X \cap (a, b)$ is nonempty (note that it's open in X), and

(ii) $X \cap (a, b) \subseteq S_N$.

Suppose $x \in X \cap (a, b)$. Then since x is a limit point of $X \cap (a, b)$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X \cap (a, b)$ converging to x. Property (ii) implies that $f^{(N)}(x_n) = 0$ for all $n \in \mathbb{N}$. Using the definition of derivative, this implies $f^{(N+1)}(x) = 0$ showing $x \in S_{N+1}$. Proceeding similarly we get,

(iii) $x \in S_m$ for all $m \geq N$ and every $x \in X \cap (a, b)$.

Consider any maximal interval $(c, d) \subseteq ((a, b) \setminus X)$. Since (c, d) lies outside X, f is a polynomial on (c, d), say of degree k. This implies that $f^{(k)} \neq 0$ on (c, d). Continuity of $f^{(k)}$ implies that in fact $f^{(k)} \neq 0$ on $[c, d]$. Since one of c or d must be in X, using property (iii) this implies $k < N$.

This along with property (iii) shows that $f^{(N)} = 0$ on the full interval (a, b). But then this means the restriction $f\restriction_{(a,b)}$ is a polynomial which contradicts property (i). □

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