Homology, Brouwer's Fixed-Point Theorem, and Invariance of Domain

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1 Motivations

Homology is a powerful tool from algebraic topology that is useful not only for characterizing topological spaces, but also for proving some important theorems that themselves have lots of applications. A classical theorem from fixed-point theory is Brouwer's fixed point theorem. It states that any continuous map from D^n to itself has at least one fixed-point, which is a point that gets mapped to itself.

One application of Brouwer's fixed point theorem is in proving the Perron-Frobenius theorem, which states that a real square matrix with non-negative entries has a maximum eigenvalue and a corresponding eigenvector with all non-negative components (MacCluer 494). The Perron-Frobenius theorem is immensely useful in real-life applications involving some process that evolves through time. For example, the theorem shows that any finite stationary Markov chain possesses a stationary distribution (MacCluer 489).

Theorem 1.1 (Brouwer fixed point theorem in 2 dimensions)**.** *Every continuous map $f : D^2 \to D^2$ has a fixed point, that is, there exists a point $x \in D^2$ with $f(x) = x$.*

Proof. Suppose there were a continuous map $f : D^2 \to D^2$ with no fixed point, then $f(x) \neq x$ for all $x \in D^2$. Then we can define a map $r : D^2 \to S^1$ by taking $r(x)$ to be the intersection of the boundary circle S^1 with the ray in $\mathbb{R}^2$ starting at $f(x)$ and passing through x. Note that r is not well-defined if f has a fixed point.

Since f is continuous, a small change in x produces a small change in $f(x)$, which amounts to a small change in the ray between these two points. Thus, r is continuous. Also, if $x \in S^1$, then any ray passing through x will intersect the circle at x, so $r(x) = x$. Therefore, r is a retraction of D^2 onto S^1.

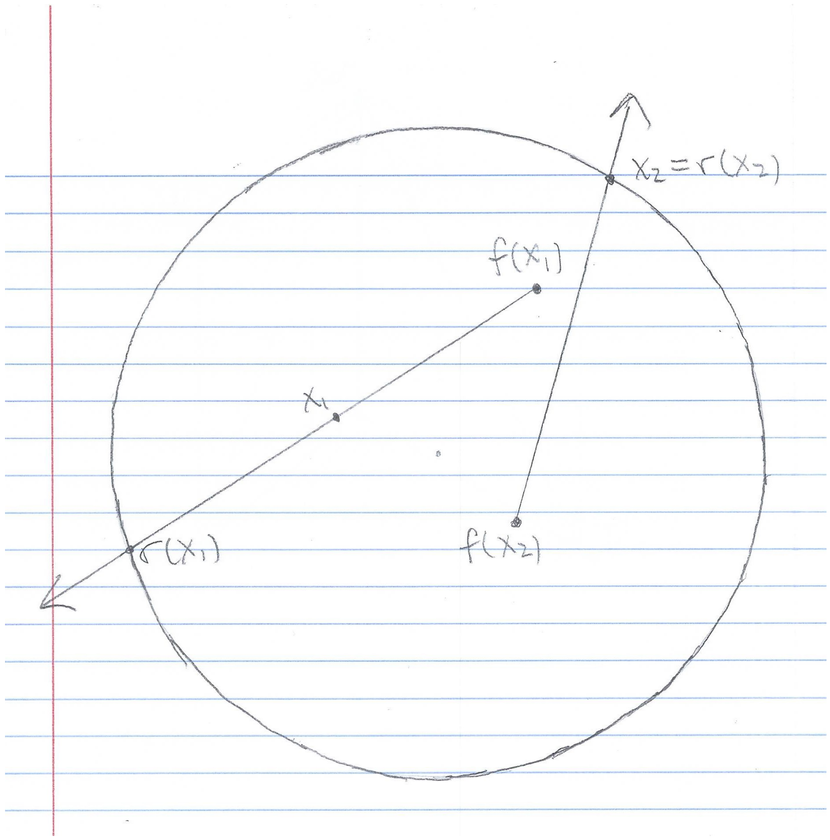

Figure 1: The retraction r

We would like to show that no such retraction exists by using the fact that the fundamental group of S^1 is nontrivial, which means that not all loops in S^1 are based-homotopic to a constant loop. Let $\gamma : S^1 \to S^1$ be a loop in S^1, then in D^2, there is a based-homotopy from γ to a constant loop at x_0 given by the linear map $F_t(s) = (1-t)\gamma(s)+tx_0$. Now since r is a retraction, r is the identity when we restrict it to S^1. Then rF_t is a based-homotopy from $rF_0 = F_0 = \gamma$ to $rF_1 = rc_{x_0} = c_{x_0}$. This means that any loop in S^1 is based-homotopic to a constant loop, which is a contradiction. $\square$

This theorem also holds for continuous maps from D^n to D^n for any n, and a proof of it uses the same ideas as above. We can still construct the retraction $r : D^n \to S^{n-1}$ using a ray in $\mathbb{R}^n$, but then we need to modify our proof a little to show that such a retraction doesn't exist.

Indeed, we proved in class that the fundamental group of S^n for $n \geq 2$ is trivial.

We could use the higher-dimensional analogoues of the fundamental group, namely the higher-homotopy groups π_n, which are defined to be the based-homotopy classes of maps from S^n. However, these are difficult to compute. Recall that we ran into a similar conundrum when proving that the interior and boundary of a topological manifold are disjoint; we were only able to show this for 2-manifolds.

Instead, we demonstrate how a different kind of group, called *homology groups*, can serve a similar function as the fundamental group, while remaining computationally accessible.

2 Axioms of Homology Groups

Before we get to homology, there are a couple group-theoretic concepts we will need to become familiar with.

Definition 2.1. *Let G, H, K be groups, and $\alpha : G \to H$, $\beta : H \to K$ be homomorphisms. Then we say that the pair of homomorphisms $G \xrightarrow{\alpha} H \xrightarrow{\beta} K$ is exact (at H) if* $\operatorname{im} \alpha = \ker \beta$.

A sequence

$$\cdots \to G_{n-1} \xrightarrow{\alpha_n} G_n \xrightarrow{\alpha_{n+1}} G_{n+1} \to \cdots$$

of homomorphisms is an exact sequence if it is exact at every G_n between a pair of homomorphisms, that is, $\operatorname{im} \alpha_n = \ker \alpha_{n+1}$.

Definition 2.2. *Let I be any nonempty index set and let G_i be a group for each $i \in I$. Then the direct sum of the groups G_i, denoted $\oplus_{i \in I} G_i$, is the subgroup of the direct product consisting of the set of elements which are the identity in all but finitely many components.*

Note that when I is finite, the direct sum is the same as the direct product.

For our goal of proving Brouwer's fixed-point theorem, it is more convenient to abstract away the details of homology groups and simply work with their properties. There are different category-theoretical systems of axioms for a *homology theory* on topological spaces and continuous maps. For example, the Eilenberg-Steenrod-Milnor axioms for homology deal with pairs of topological spaces and relative homology (Bredon 183).

Here, we will present a less general set of properties, which we will consider axioms, that can be derived from these more general axioms. In a later section, we will actually construct homology groups and prove that they satisfy these axioms. Notationally, we will use $\widetilde{h}_n$ to refer to the abstract reduced homology groups in the axioms, and we will use $\widetilde{H}_n$ to refer to the (reduced) singular homology groups of our construction.

For any topological space X and $n \in \mathbb{Z}_{\geq 0}$, we can assign an abelian group $\widetilde{h}_n(X)$.

Axiom 2.3. *If Y is also a topological space with $f : X \to Y$ continuous, then we can assign to f, n a homomorphism $f_* : \widetilde{h}_n(X) \to \widetilde{h}_n(Y)$ satisfying $(fg)_* = f_* g_*$ and $(\mathrm{id}_X)_* = \mathrm{id}_{\widetilde{h}_n(X)}$.*

Axiom 2.4 (Homotopy axiom). *Let X and Y be topological spaces, and $f, g : X \to Y$ continuous maps such that f is homotopic to g. Then $f_* = g_*$.*

Definition 2.5. *Let X be a space, and $A \subset X$, then we say that (X, A) is a good pair if A is nonempty, closed, and is a deformation-retract of some open $U \subset X$.*

Axiom 2.6 (Exactness axiom). *Let X be a space, and (X, A) be a good pair. Then there are boundary homomorphisms $\partial' : \widetilde{h}_n(X/A) \to \widetilde{h}_{n-1}(A)$ where we get the following long exact sequence*

$$\cdots \xrightarrow{\partial'} \widetilde{h}_n(A) \xrightarrow{i_*} \widetilde{h}_n(X) \xrightarrow{p_*} \widetilde{h}_n(X/A) \xrightarrow{\partial'} \widetilde{h}_{n-1}(A) \xrightarrow{i_*} \cdots$$

where i is the inclusion map and p is the projection map.

Axiom 2.7 (Dimension axiom). *For a one-point space $*$, $\widetilde{h}_n(*) = 0$ is the trivial group for all $n \geq 0$.*

The next axiom will not be needed in any of our proofs, but we include it here for completeness. We will omit the proof and refer the details to Hatcher (126) (however, note that for unreduced homology, the wedge sum must be replaced with the disjoint union).

Axiom 2.8 (Additivity axiom). *For a wedge sum $X = \bigvee_\alpha X_\alpha$, where each (X, X_α) are good pairs, and inclusions $i_\alpha : X_\alpha \hookrightarrow X$, the direct sum map $\oplus_\alpha i_{\alpha *} : \oplus_\alpha \widetilde{h}_n(X_\alpha) \to \widetilde{h}_n(X)$ is an isomorphism for each n.*

Axiom 2.9 (Coefficient axiom). *$\widetilde{h}_0(S^0) = \mathbb{Z}$ and $\widetilde{h}_n(S^0) = 0$ for all $n \neq 0$.*

A theory with only axioms 2.3, 2.4, 2.6, and 2.8 is called a generalized homology theory, and can have groups for $n < 0$ that are nontrivial. Adding the dimension axiom almost uniquely determines the homology theory, and the homology groups for essentially all spaces are uniquely determined once we add the coefficient axiom. We could have chosen the coefficient group $\widetilde{h}_0(S^0)$ to be any abelian group, but it is conventional to have it be $\mathbb{Z}$. We will take all of these axioms together so we can talk about, up to isomorphism, the one value of $\widetilde{H}_n(X)$ for any space X.

This next proposition and its proof are exactly analogous to the case of the fundamental group.

Proposition 2.10. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : \widetilde{h}_n(X) \to \widetilde{h}_n(Y)$ are isomorphisms for all n.*

Proof. If $f : X \to Y$ is a homotopy equivalence with homotopy inverse g, then gf is homotopic to id_X and fg is homotopic to id_Y. From the Homotopy axiom,

$$(gf)_* = g_* f_* = (\mathrm{id}_X)_* = \mathrm{id}_{\widetilde{h}_n(X)},$$

and

$$(fg)_* = f_* g_* = (\mathrm{id}_Y)_* = \mathrm{id}_{\widetilde{h}_n(Y)},$$

therefore $f_* : \widetilde{h}_n(X) \to \widetilde{h}_n(Y)$ are isomorphisms with inverses g_*. $\square$

3 Examples

Using these properties, we can compute homology groups for some important spaces. As is the case for the fundamental group, the homology groups of any contractible space are trivial, as illustrated by the following example.

Example 3.1. $\widetilde{H}_i(D^n) = 0$ *for all* $i \geq 0$.

Proof. $X = D^n$ is contractible, so by the homotopy axiom, $\widetilde{H}_i(D^n) = \widetilde{H}_i(*)$ for $*$ a one-point space. Thus, by the dimension axiom, $\widetilde{H}_i(D^n) = 0$ is trivial for any $i \geq 0$. $\square$

As mentioned in the introduction, we will need to find a nontrivial homology group of S^n in order to prove Brouwer's fixed point theorem. To do this, we make use of the exactness axiom. Specifically, we will use that $\widetilde{H}_i(D^n) = 0$ and $D^n/S^{n-1} = S^n$ to come up with a bunch of isomorphisms, and then we will use induction to push from S^0 to S^n.

Example 3.2. $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ *and* $\widetilde{H}_i(S^n) = 0$ *for* $i \neq n$.

Proof. For $n > 0$, consider the pair (D^n, S^{n-1}). This is a good pair because S^{n-1} is nonempty, closed, and we can take U to be the annulus $U = D^n \setminus \overline{B_{1/2}(0)}$, which is open in D^n. Then $S^{n-1} \subset U$ and U deformation retracts to S^{n-1} by a straight-line homotopy on radial lines. Also, $D^n/S^{n-1} \cong S^n$.

By the exactness axiom, we obtain the following exact sequence:

$$\widetilde{H}_i(D^n) \xrightarrow{p_*} \widetilde{H}_i(D^n/S^{n-1}) \xrightarrow{\partial} \widetilde{H}_{i-1}(S^{n-1}) \xrightarrow{i_*} \widetilde{H}_{i-1}(D^n).$$

We know that $\widetilde{H}_i(D^n)$ is trivial for all i, and $D^n/S^{n-1} \cong S^n$ means

$$\widetilde{H}_i(D^n/S^{n-1}) \cong \widetilde{H}_i(S^n)$$

by Proposition 2.10. By exactness of the sequence, we see that $\ker \partial = \operatorname{im} p_* = 0$ since $\widetilde{H}_i(D^n) = 0$, and also $\operatorname{im} \partial = \ker i_*$ is the whole group since $\widetilde{H}_{i-1}(D^n) = 0$. Therefore,

$$\partial : \widetilde{H}_i(D^n/S^{n-1}) \to \widetilde{H}_{i-1}(S^{n-1})$$

is an isomorphism, which means $\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$ for all $n \geq 1$.

Using that $\widetilde{H}_0(S^0) = \mathbb{Z}$ and $\widetilde{H}_i(S^0) = 0$ for $i \neq 0$, we see by induction that $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{H}_i(S^n) = 0$ for all $i \neq n$. $\square$

4 Applications

Now that we have some calculations, we are ready to prove Brouwer's fixed-point theorem. As mentioned in the introduction, the proof will follow the same outline as the 2-dimensional case, except we replace the fundamental group with homology groups.

Theorem 4.1 (Brouwer's Fixed-Point Theorem). *There is no retraction from D^n to its boundary S^{n-1}. Thus, every continuous map $f : D^n \to D^n$ has a fixed point.*

Proof. Suppose there were a continuous map $f : D^n \to D^n$ with no fixed point, then $f(x) \neq x$ for all $x \in D^n$. Then we can define a map $r : D^n \to S^{n-1}$ by taking $r(x)$ to be the intersection of the boundary circle S^{n-1} with the ray in $\mathbb{R}^n$ starting at $f(x)$ and passing through x. As before, this is continuous and is a retraction.

Now suppose we have a retraction $r : D^n \to S^{n-1}$, then $ri = \mathrm{id}_{S^{n-1}}$ for $i : S^{n-1} \to D^n$ the inclusion map. We consider the induced maps of i and r on homology groups:

$$\widetilde{H}_{n-1}(S^{n-1}) \xrightarrow{i_*} \widetilde{H}_{n-1}(D^n) \xrightarrow{r_*} \widetilde{H}_{n-1}(S^{n-1}).$$

Since $r_* i_* = (ri)_* = \mathrm{id}_*$, this means that $r_* i_*$ is the identity map on $\widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$. However, since $\widetilde{H}_{n-1}(D^n) = 0$, r_* and i_* are both the trivial homomorphism, so $r_* i_*$ is the trivial homomorphism. This is a contradiction since $\mathbb{Z}$ is nontrivial. $\square$

We can also address the problem we mentioned in the introduction about proving that the interior and boundary of a topological manifold are disjoint. The proof of this is almost exactly the same as the proof for surfaces, except we replace the fundamental groups with homology groups.

Theorem 4.2. *Let M be a topological n-manifold with boundary, then the interior of M and the boundary of M are disjoint.*

Proof. Suppose that $x \in M$ is both in the interior and the boundary of M. Then there are neighborhoods H, U of x with homeomorphisms $\phi : U \to \mathbb{R}^n$ and $\psi : H \to \mathbb{R}^{n-1} \times [0, \infty)$; we can arrange for $\phi(x) = \psi(x) = 0$.

Let $H_m = \psi^{-1}(S_{1/m}(0))$ where $S_{1/m}(0) = \{x \in \mathbb{R}^{n-1} \times [0, \infty) \mid ||x|| < 1/m\}$. Then since $H \cap U \subset H$ is open and contains x, and ψ is a homeomorphism, $\psi(H \cap U) \subset \mathbb{R}^{n-1} \times [0, \infty)$ is an open subset containing 0. Thus, for some m large enough, $H_m \subset U$.

In the same way, let $U_k = \phi^{-1}(B_{1/k}(0))$, then since $H_m \cap U \subset U$ is open and contains x, and ϕ is a homeomorphism, $\phi(H_m \cap U) \subset \mathbb{R}^n$ is an open subset containing 0. Thus, for some k large enough, $U_k \subset H_m \subset U$.

We consider the homomorphisms induced by the inclusion maps

$$\widetilde{H}_{n-1}(U_k \setminus \{x\}) \xrightarrow{i_{1*}} \widetilde{H}_{n-1}(H_m \setminus \{x\}) \xrightarrow{i_{2*}} \widetilde{H}_{n-1}(U \setminus \{x\}).$$

By construction, $U_k \setminus \{x\}$ and $U \setminus \{x\}$ are homeomorphic to $\mathbb{R}^n \setminus \{0\}$, which is homotopy equivalent to S^{n-1}. Thus, these spaces have homotopy groups isomorphic to $\mathbb{Z}$. On the other hand, $H_m \setminus \{x\}$ is homeomorphic to $\mathbb{R}^{n-1} \times [0, \infty) \setminus \{0\}$, which is contractible (e.g. by a straight line homotopy to $(0, 0, \dots, 1)$). Hence, $\widetilde{H}_{n-1}(H_m \setminus \{x\}) = 0$.

Clearly, the composition of the inclusion maps $i_2 i_1$ is an inclusion from U_k to U, which induces the identity homomorphism. However, i_{1*} and i_{2*} are zero maps since $\widetilde{H}_{n-1}(H_m \setminus \{x\}) = 0$. This contradicts the axiom about compositions of induced homomorphisms. Thus, there is no point that is in both the interior and boundary of M.

□

Another theorem that can be proven using homology is the invariance of domain theorem, which is important in the study of topological manifolds. In order to prove this theorem, we will need a couple of technical lemmas. We will provide sketches of their proofs (deferring the proof of the Mayer-Vietoris sequence to Section 5), but complete proofs can be found in Hatcher (149-50, 169-70).

Theorem 4.3 (Mayer-Vietoris sequence)**.** *Let $A, B \subset X$ such that X is the union of the interiors of A and B, then there is an exact sequence*

$$\cdots \to \widetilde{H}_n(A \cap B) \xrightarrow{\Phi} \widetilde{H}_n(A) \oplus \widetilde{H}_n(B) \xrightarrow{\Psi} \widetilde{H}_n(A \cup B) \xrightarrow{\partial} \widetilde{H}_{n-1}(A \cap B) \to \cdots .$$

Lemma 4.4. *For any topological embedding $f : D^k \to S^n$, $\widetilde{H}_i(S^n \setminus f(D^k)) = 0$ for all i.*

For any topological embedding $f : S^k \to S^n$ with $k < n$, $\widetilde{H}_i(S^n \setminus f(S^k))$ is $\mathbb{Z}$ for $i = n - k - 1$ and 0 otherwise.

Proof. The proof of the first part uses induction on k. For $k = 0$, $S^n \setminus f(D^0)$ is the sphere with a point deleted, which we know is homeomorphic to $\mathbb{R}^n$, so $\widetilde{H}_i(S^n \setminus f(D^0)) = 0$. For the inductive step, we replace D^k by the cube I^k, where $I = [0, 1]$, which allows us to decompose $I^k = (I^{k-1} \times [0, 1/2]) \cup (I^{k-1} \times [1/2, 1])$.

This decomposition gives us a natural way to construct a Mayer-Vietoris sequence. Take $A = S^n \setminus f(I^{k-1} \times [0, 1/2])$ and $B = S^n \setminus f(I^{k-1} \times [1/2, 1])$, then $A \cap B = S^n \setminus f(I^k)$ and $A \cup B = S^n \setminus f(I^{k-1} \times \{1/2\})$. Clearly, we can consider h restricted to $I^{k-1} \times \{1/2\}$ as an embedding from I^{k-1}, so by the inductive hypothesis $\widetilde{H}_i(A \cup B) = 0$ for all i. Thus, we obtain the following portion of the Mayer-Vietoris sequence

$$\widetilde{H}_{i+1}(A \cup B) \to \widetilde{H}_i(A \cap B) \xrightarrow{\Phi} \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to \widetilde{H}_i(A \cup B)$$

for all i. Since the terms on either end are 0, we get that $\Phi : \widetilde{H}_i(S^n \setminus f(I^k)) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B)$ is an isomorphism.

Suppose that we have some nontrivial element of $\widetilde{H}_i(S^n \setminus f(I^k))$, then it must get mapped to something nontrivial in at least one of $\widetilde{H}_i(A)$ or $\widetilde{H}_i(B)$. Without loss of generality, assume it is A. Notice that $I^{k-1} \times [0, 1/2]$ is just I^k but with a shorter interval in the last coordinate, which means we can repeat the same argument as above on A.

Continuing this process, we get a sequence of intervals $I = I_0 \supset I_1 \supset I_2 \supset \cdots$ where I_j has length 2^{-j}, and α maps to some $\alpha_j \neq 0$ in $\widetilde{H}_i(S^n \setminus f(I^{k-1} \times I_j))$. Essentially, we take this process to infinity, and since the intervals I_j are nonempty and compact,

$$\bigcap_{j=0}^{\infty} I_j = \{x\}$$

for some point $x \in I$. It takes some work to make the idea of homology groups for limits of spaces precise, but after working things out, we get α mapping to a nontrivial element in $\widetilde{H}_i(S^n \setminus f(I^{k-1} \times \{x\}))$, which is trivial by the inductive hypothesis. This is a contradiction, so there is no nontrivial element of $\widetilde{H}_i(S^n \setminus f(I^k))$, i.e. this group is trivial.

The proof of the second part also uses induction on k. For $k = 0$, $S^n \setminus f(S^0)$ is the sphere with two points deleted, which is homeomorphic to $S^{n-1} \times \mathbb{R}$, which is homotopy-equivalent to S^{n-1}. Thus, the statement follows from our calculations of the homology groups of S^n.

For the inductive step, we decompose S^k into two closed hemispheres D^k_+ and D^k_-, which intersect in a great circle S^{k-1}. Each hemisphere is homeomorphic to a disc D^k by flattening. Thus, we can take $A = S^n \setminus f(D^k_+)$ and $B = S^n \setminus f(D^k_-)$, then by the first part, $\widetilde{H}_i(A)$ and $\widetilde{H}_i(B)$ are trivial. Therefore, the Mayer-Vietoris sequence gives isomorphisms

$$\widetilde{H}_i(S^n \setminus f(S^k)) \cong \widetilde{H}_{i+1}(S^n \setminus f(S^{k-1})).$$

Hence, by induction, $\widetilde{H}_i(S^n \setminus f(S^k))$ is isomorphic to $\mathbb{Z}$ when $i = n - k - 1$, and is trivial otherwise.

□

The first part of this lemma tells us that $S^n \setminus f(D^k)$ is path-connected. The second part of this lemma is known as the Generalized Jordan Curve Theorem, and it generalizes the notion that any loop in the plane has an "inside" and an "outside". As a special case of this theorem, if we have an embedding $f : S^{n-1} \to S^n$, then $\widetilde{H}_0(S^n \setminus f(S^{n-1})) \cong \mathbb{Z}$, so the unreduced homology group $H_0(S^n \setminus f(S^{n-1})) \cong \mathbb{Z} \oplus \mathbb{Z}$. One can show that this means $S^n \setminus f(S^{n-1})$ consists of exactly two path components (see Bredon 172).

Theorem 4.5 (Invariance of Domain)**.** *Let M be a topological n-manifold and $f : M \to \mathbb{R}^n$ a continuous injection, then f is an open mapping.*

Proof. Let $x \in M$, then we can find an open neighborhood U of x, and a disk $D^n \subset U$ with center x. We want to show that $f(D^n \setminus \partial D^n) \subset f(U)$ is open in $\mathbb{R}^n$, which would mean that $f(U)$ is a neighborhood of $f(x)$. Considering the one-point compactification of $\mathbb{R}^n$, which is homeomorphic to S^n, it suffices to show that $f(D^n \setminus \partial D^n)$ is open in S^n.

Because f is a continuous injection, its restrictions to D^n and $\partial D^n = S^{n-1}$ are embeddings, so by the previous lemma, $S^n \setminus f(\partial D^n)$ has two path-components. We can figure out exactly what these path-components are: they are $f(D^n \setminus \partial D^n)$ and $S^n \setminus f(D^n)$. The first space is homeomorphic to $D^n \setminus \partial D^n$, which is clearly path-connected, the second is path-connected by the lemma, and these spaces are disjoint.

Since $S^n \setminus f(\partial D^n)$ is open in S^n, its path-components are exactly its connected components. For a space with finitely many connected components, each connected component is open, hence $f(D^n \setminus \partial D^n)$ is open in $S^n \setminus f(\partial D^n)$, so it is open in S^n. □

Theorem 4.6. *$\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $n = m$.*

Proof. Obviously, $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$. Suppose for $m < n$ that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism. We have the embedding $g : \mathbb{R}^m \to \mathbb{R}^n$ given by $g(x) = x + 0e_{m+1} + \cdots + 0e_n$ for e_i the standard basis vectors. Clearly, $g(\mathbb{R}^m)$ is not open in $\mathbb{R}^n$.

We know that the composition $fg : \mathbb{R}^m \to \mathbb{R}^m$ is an embedding, so it is an open map. However, $(fg)(\mathbb{R}^m)$ is not open, and $\mathbb{R}^m$ is open in itself, which is a contradiction. □

In fact, it is true that if a topological m-manifold is homeomorphic to a topological n-manifold, then $n = m$. All we need to do is to construct an embedding from $\mathbb{R}^m$ to any n-manifold such that the image of $\mathbb{R}^m$ is not open, and to generalize the invariance of domain theorem to work when the codomain is a topological n-manifold instead of $\mathbb{R}^n$. The details of this can be found in Bredon (235).

5 Definition of Homology

Now is the part where we must come back down to earth and actually construct a homology theory satisfying the axioms. This construction is called *singular homology*.

Definition 5.1. *An n-simplex is the smallest convex set in $\mathbb{R}^n$ containing a collection of $n+1$ points $v_0, \ldots, v_n$ such that $v_1 - v_0, \ldots, v_n - v_0$ form a linearly independed set. This is independent of the choice of v_0. We denote this simplex by $[v_0, \ldots, v_n]$.*

Definition 5.2. *The standard n-simplex Δ^n is the set*

$$\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, t_i \geq 0\}.$$

Visually, this is just the n-dimensional analog of the triangle, where the vertices are the unit vectors e_i. The specific ordering of the vertices matters when we define homology.

Definition 5.3. *A singular n-simplex in a space X is a continuous map $\sigma : \Delta^n \to X$.*

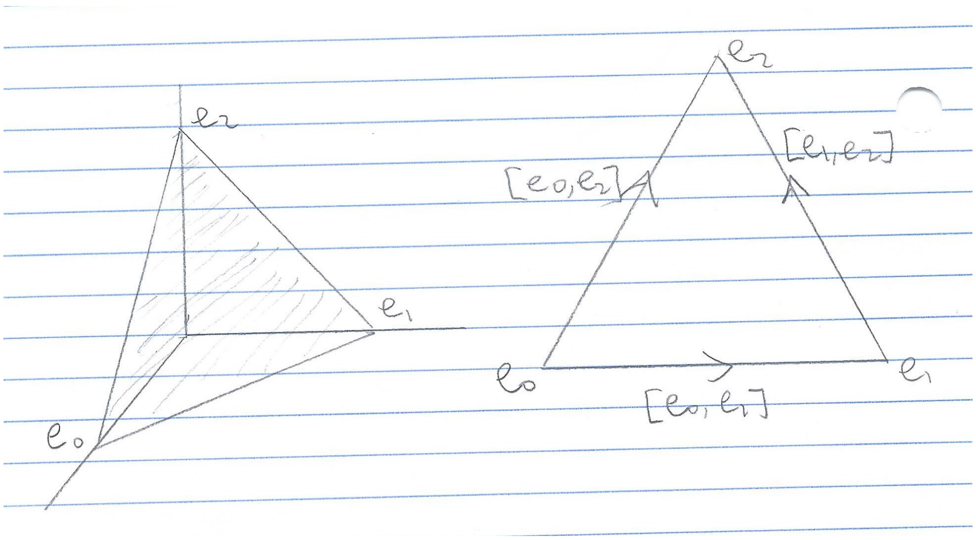

Figure 2: The standard 2-simplex and its faces

Definition 5.4. *We define $C_n(X)$ to be the free abelian group with basis the set of singular n-simplices in X. Elements of $C_n(X)$ are called n-chains, and each n-chain is just a finite formal sum $\sum_i n_i\sigma_i$ for $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$.*

Definition 5.5. *If σ is a singular n-simplex, and i is an integer with $0 \leq i \leq n$, then we define the i-th face of σ, denoted σ_i, to be the singular $(n-1)$-simplex in X given by*

$$\sigma_i(t_0, \dots, t_{n-1}) = \sigma(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}).$$

Essentially what is happening here is that we take a $(n-1)$-simplex, embed it into an n-simplex as its i-th face, and then embed that into X through the map σ.

Definition 5.6. *A boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by the formula*

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma_i.$$

This is a homomorphism.

Specifically, if we identify $[v_0, \dots, v_n]$ as Δ^n, then we can write this formula as

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \mid [v_0, \dots, \hat{v}_i, \dots, v_n].$$

The hat over v_i means that we omit this vertex, so $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ is an $(n-1)$-simplex, and is the i-th face of Δ^n. Thus, the formula means that we sum over the restrictions of σ to the i-th face of Δ^n. The alternating signs are to account for the orientations of faces. Sometimes, when the index n is clear from context or is unimportant, we omit the subscript and just write ∂ for ∂_n.

Proposition 5.7. *Let $\partial_n : C_n(X) \to C_{n-1}(X)$ and $\partial_{n+1} : C_{n+1}(X) \to C_n(X)$ be boundary homomorphisms. Then their composition $\partial_n \partial_{n+1} = 0$.*

The proof of this is just computations using the formula, so I will leave the details to Hatcher (108).

Definition 5.8. *The singular homology group $H_n(X)$ is defined to be* $\ker \partial_n / \operatorname{im} \partial_{n+1}$.

Basically, $C_n(X)$ is too big of a group to be useful, so we restrict ourselves to cycles, which are n-chains with 0 boundary, modulo the relationship where two cycles are equivalent if taking the difference of the two gives us the boundary of an $n+1$-chain. Historically, these terms come from Poincaré's work in developing the concept of homology; our definitions formalize his ideas by making his notions of "cycle" and "boundary" algebraic (Dieudonné 17).

The point of these definitions is that we want to study a space X by seeing what sort of ways there are to map from the standard n-simplex to X. The reason why working with simplices is easier than working with n-spheres, as in the case of higher homotopy groups, is that we have this nice notion of the boundary of an n-simplex, which is itself a collection of $(n-1)$-simplices. This boundary map and the ability to add and subtract singular simplices means that we can effectively break up and reassemble a map from Δ^n in any order.

Proposition 5.9. *If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \cong \mathbb{Z}$.*

Proof. For each n, there is only one singular n-simplex σ_n, which sends every point to the single point in X. Thus, $C_n(X)$ is the free abelian group generated by one element, so it is isomorphic to $\mathbb{Z}$.

Now examining the boundary map ∂_n, we see that $\partial_n(\sigma)$ is an alternating sum of $n+1$ terms which are all equal. Therefore, $\partial_n = 0$ if n is odd, and $\partial_n(\sigma_n) = \sigma_{n-1}$ if n is even and greater than 0. Therefore, we have a sequence of homomorphisms

$$\cdots \to \mathbb{Z} \xrightarrow{\partial_4} \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

where the boundary maps alternate between isomorphisms and trivial maps.

It is clear that $\ker \partial_0 \cong \mathbb{Z}$ while $\operatorname{im} \partial_1 = 0$, hence $H_0(X) \cong \mathbb{Z}$. Otherwise, for n odd, $\ker \partial_n = \mathbb{Z}$ while $\operatorname{im} \partial_{n+1} = \mathbb{Z}$, and if n is even, $\ker \partial_n = 0$. Thus, $H_n(X) = 0$ for $n > 0$. □

It is convenient to make the homology groups of a point trivial in all dimensions, and we can do that by using reduced homology groups $\widetilde{H}_n(X)$. For X nonempty, these are constructed by adding a copy of $\mathbb{Z}$ in the sequence

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i$. This does not change the homology groups for $n \geq 1$, but for $n = 0$, we simply have $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$.

Example 5.10. *Let's try to understand all these notions for $X = S^1$. First of all, Δ^0 is just a single point, so a 0-simplex σ in S^1 is just a point in S^1. Thus, a 0-chain is just a formal sum $\sum_x n_x x$ where $n_x = 0$ except for a finite number of points $x \in S^1$. By convention, $\partial_0 = 0$, so any 0-chain is a 0-cycle.*

Next, Δ^1 is a line segment $[e_0, e_1]$, so a 1-simplex σ can be considered a path in S^1. The boundary map $\partial_1 \sigma = \sigma \mid [e_0] - \sigma \mid [e_1]$ is just the difference of the endpoints of σ. Thus, any loop in S^1 is a 1-cycle. This also shows that any two 0-chains differ by a boundary, since we can take $\sigma : \Delta^1 \to S^1$ to be the path from one point to the other point.

Let's consider σ_1 to be the loop starting at 1 that winds once counterclockwise around the circle, σ_2 the half loop that goes counterclockwise from 1 to -1, and σ_3 the half loop that goes counterclockwise from -1 to 1. Then σ_1 is a 1-cycle, and $\sigma_2 + \sigma_3$ is also a 1-cycle because

$$\begin{aligned}\partial_1(\sigma_2 + \sigma_3) &= (\sigma_2 + \sigma_3) \mid [e_0] - (\sigma_2 + \sigma_3) \mid [e_1] \\ &= (1 + (-1)) - ((-1) + 1) \\ &= (1 - 1) + ((-1) - (-1)) = 0.\end{aligned}$$

We claim that σ_1 and $\sigma_2 + \sigma_3$ differ by a boundary: first, consider the singular 2-simplex $\sigma_4 : \Delta^2 \to S^1$ which is σ_2 on the edge $[e_0, e_1]$, σ_3 on the edge $[e_1, e_2]$ and extended to the rest of Δ^2 by having σ_4 be constant on the lines perpendicular to $[e_0, e_2]$. Then we get the concatenation $\sigma_2 * \sigma_3$ on edge $[e_0, e_2]$, which means

$$\partial_2 \sigma_4 = \sigma_2 - (\sigma_2 * \sigma_3) + \sigma_3.$$

Hence, $\sigma_2 + \sigma_3$ and $\sigma_2 * \sigma_3$ differ by a boundary. But notice that $\sigma_2 * \sigma_3$ is equal to σ_1, so σ_1 and $\sigma_2 + \sigma_3$ represent the same element in $H_1(S^1)$.

We still need to talk about induced homomorphisms between homology groups. For a map $f : X \to Y$, we define an induced homomorphism $f_\# : C_n(X) \to C_n(Y)$ by composition $f_\#(\sigma) = f\sigma$, then extending linear to get the formula

$$f_\#(\sum_i n_i \sigma_i) = \sum_i n_i f \sigma_i.$$

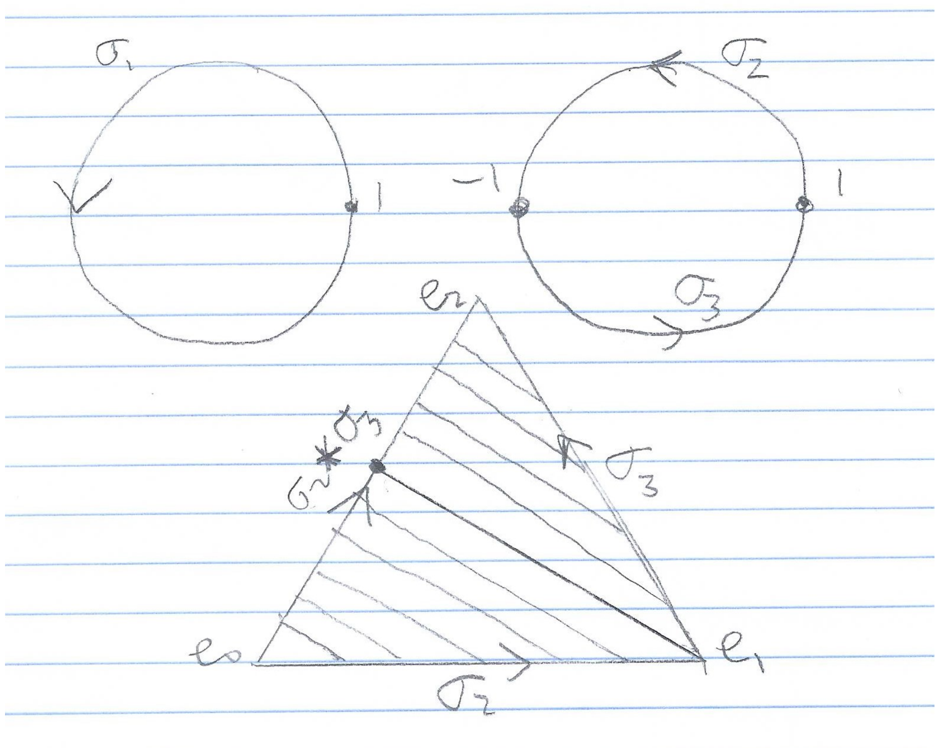

Figure 3: $\sigma_1, \sigma_2, \sigma_3$, and σ_4

We can verify that $f_\# \partial = \partial f_\#$ by the formulae.

If σ is a cycle, that is $\partial\sigma = 0$, then $\partial(f_\#\sigma) = f_\#(\partial\sigma) = 0$, so $f_\#\sigma$ is a cycle. If σ is a boundary, meaning $\sigma = \partial\beta$ for some β, then $f_\#(\sigma) = f_\#(\partial\beta) = \partial(f_\#\beta)$, which means $f_\#\sigma$ is a boundary. Therefore, $f_\#$ preserves cycles and boundaries, so $f_\#$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$, and likewise for reduced homology groups $f_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$.

From this, we can verify that $(fg)_* = f_* g_*$ from associativity of compositions $\Delta^n \xrightarrow{\sigma} X \xrightarrow{g} Y \xrightarrow{f} Z$. Also, it is clear that the identity map induces the identity homomorphism on homology groups $\mathrm{id}_* = \mathrm{id}$.

As one can see, the definition of singular homology is quite technical, and unfortunately, the proofs of the axioms of homology are no less technical. We shall sketch the proofs and leave the details to be found in Hatcher.

Theorem 5.11. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$.*

Proof. Let $F : X \times [0,1] \to Y$ be a homotopy from f to g, then for a singular simplex $\sigma : \Delta^n \to X$ we can form a prism operator $P : C_n(X) \to C_{n+1}(Y)$ that satisfies the relation

$$(*) \qquad \partial P = g_\# - f_\# - P\partial.$$

What this prism operator does is it takes a prism $\Delta^n \times I$ and breaks it up into $(n+1)$-simplices. The picture to keep in mind is we have one copy of Δ^n given by $\Delta^n \times \{0\}$ with vertices $[v_0, \ldots, v_n]$, and another copy of Δ^n above it given by $\Delta^n \times \{1\}$ with vertices $[w_0, \ldots, w_n]$.

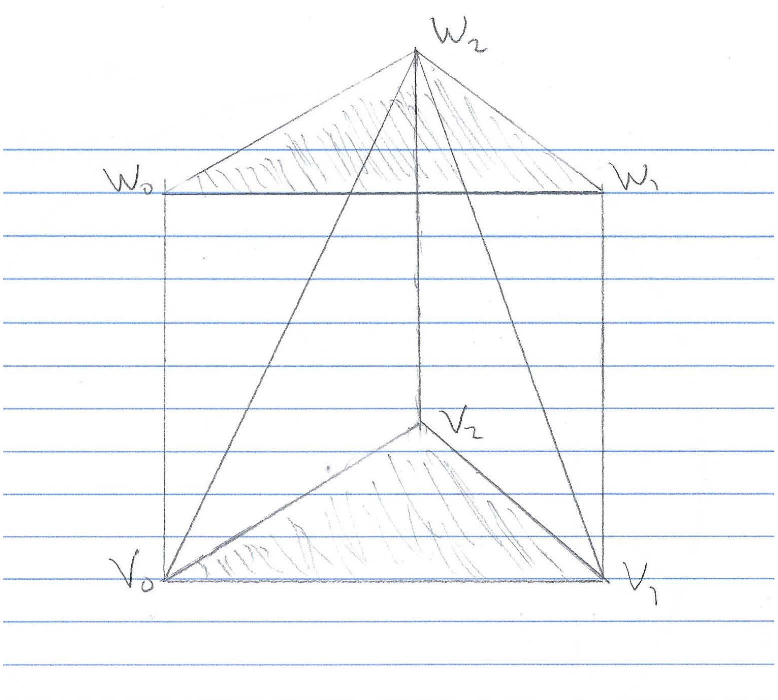

Figure 4: The prism $\Delta^2 \times I$ and the 3-simplex $[v_0, v_1, v_2, w_2]$

If we move one vertex of $[v_0, \ldots, v_n]$ up to get $[v_0, \ldots, v_{n-1}, w_n]$, then the region between these two n-simplices is an $(n+1)$-simplex given by $[v_0, \ldots, v_{n-1}, v_n, w_n]$. The prism operator repeats this process of creating $(n+1)$-simplices and pushing them through F to get the formula

$$P(\sigma) = \sum_i (-1)^i F(\sigma \times \mathrm{id}_I) \mid [v_0, \ldots, v_i, w_i, \ldots, w_n].$$

Proving the relation (*) amounts to some calculation with the formula, which we leave to Hatcher (112).

Given this prism operator, we see that if $\alpha \in C_n(X)$ is a cycle, then

$$g_{\#}(\alpha) - f_{\#}(\alpha) = \partial P(\alpha) + P\partial(\alpha) = \partial P(\alpha)$$

since $\partial\alpha = 0$. This means that $g_{\#}(\alpha) - f_{\#}(\alpha)$ is a boundary. Hence, when we quotient by $\operatorname{im}\partial_{n+1}$, we consider $f_{\#}(\alpha)$ and $g_{\#}(\alpha)$ to be the same element, so $f_{\#}(\alpha)$ and $g_{\#}(\alpha)$ determine the same homology class. Therefore, $f_* = g_*$. ☐

Theorem 5.12. *Let (X, A) be a good pair, then there is an exact sequence*

$$\cdots \xrightarrow{\partial} \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{p_*} \widetilde{H}_n(X/A) \xrightarrow{\partial'} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \cdots .$$

Proof. For each n, there is a short exact sequence

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{p} C_n(X/A) \to 0,$$

where i and p are the inclusion and projection maps, respectively. We also have the boundary map ∂ to pass from C_n of these spaces to C_{n-1}. Stacking these sequences on top of each other, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & C_{n+1}(A) & \xrightarrow{i} & C_{n+1}(X) & \xrightarrow{p} & C_{n+1}(X/A) & & \to 0 \\
 & & \big\downarrow \partial & & \big\downarrow \partial & & \big\downarrow \partial & & \\
0 \to & & C_n(A) & \xrightarrow{i} & C_n(X) & \xrightarrow{p} & C_n(X/A) & & \to 0 \\
 & & \big\downarrow \partial & & \big\downarrow \partial & & \big\downarrow \partial & & \\
0 \to & & C_{n-1}(A) & \xrightarrow{i} & C_{n-1}(X) & \xrightarrow{p} & C_{n-1}(X/A) & & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

This induces a diagram for the homology groups. In order to get the exact sequence we want, we just need to construct a map $\partial' : H_n(C) \to H_{n-1}(A)$ that fits between the inclusion and projection maps.

Suppose $c \in C_n(X/A)$ is a cycle, then $c = p(b)$ for some $b \in C_n(X)$. We know that $\partial b \in C_{n-1}(X)$ is in $\ker p$ since

$$p(\partial b) = \partial p(b) = \partial c = 0.$$

Therefore, since $\ker p = \operatorname{im} i$, $\partial b = i(a)$ for some $a \in C_{n-1}(A)$. Then,

$$i(\partial a) = \partial i(a) = \partial\partial b = 0,$$

which means $\partial a = 0$ since i is injective. Finally, we define $\partial' : H_n(C) \to H_{n-1}(A)$ by $\partial'[c] = [a]$. We leave the details to Hatcher to check that this map is well-defined and fits into the exact sequence (116-17). $\square$

This technique of forming short exact sequences of the C_n groups, then connecting different levels of the corresponding sequences for homology groups, is a common process in homological algebra. One can refer to Vick for a discussion of how to generalize this process (17-19). We will use it once more to prove the existence of Mayer-Vietoris sequences.

Proof of Mayer-Vietoris sequences. Let $C_n(A+B)$ be the subgroup of $C_n(X)$ consisting of sums of elements from $C_n(A)$ and $C_n(B)$. The boundary map $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$ and likewise for B, thus $\partial(C_n(A+B)) = C_{n-1}(A+B)$. We can form the short exact sequences

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0,$$

where $\phi(x) = (x, -x)$ and $\psi(x, y) = x + y$. See Hatcher for a verification that this sequence is exact (150). Like the case for the exactness axiom, these exact sequences induce exact sequences for homology groups.

It remains to show that $H_n(A+B)$ is isomorphic to $H_n(X)$, and to construct a boundary map $\partial' : H_n(X) \to H_{n-1}(A \cap B)$. The isomorphism follows from Proposition 2.21 of Hatcher (149). As for the boundary map, if we have some $\alpha \in H_n(X)$ represented by a cycle $z \in C_n(X)$, then we can somehow decompose $z = x + y$ for x, y chains in A and B, respectively. Then $\partial(x+y) = 0$, which means $\partial x = -\partial y$, hence we just let $\partial \alpha = [\partial x] = [-\partial y]$ (Hatcher 150).

$\square$

6 References

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