Random Walk

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1 The Ballot Theorem

Suppose Candidate A gets m votes and Candidate B gets n votes with $m > n$. We want to see how many ways we can count the votes such that A is always in the lead. Votes will be shown with a lattice path from $(0, 0)$ to $(m + n, m - n)$.

Theorem 1.1. *Let n and m be positive integers. There are exactly*

$$\binom{n}{\frac{n+m}{2}}$$

paths $((0, 0), (1, y_1), (2, y_2), \ldots, (n - 1, y_{n-1}), (n, m))$ from the origin to the point (n, m), $n, m > 0$, $|y_k - y_{k+1}| = 1$, such that $y_1, y_2, \ldots, y_n > 0$.

Lemma 1.2 (The Reflection Principle)**.** *For $n, m, y > 0$, the number of paths from $(0, y)$ to (n, m) which touch the t-axis equals the number of paths from $(0, -y)$ to (n, m).*

Proof. Consider a path $((0, y), (1, y_1), \ldots, (n-1, y_{n-1}), (n, m))$ from $(0, y)$ to (n, m) which has at least one vertex on the t-axis. Let r be the time of the first visit to zero; that is r satisfies $y_i > 0, \ldots, y_{r-1} > 0, y_r = 0$. Then $((0, -y), (1, -y_1), \ldots, (r - 1, -y_{r-1}), (r, y_r), \ldots, (n, m))$ is a path leading from $(0, -y)$ to (n, m). This map is a bijection and the Reflection Principle follows. □

Lemma 1.3. *For $m, n \in \mathbb{N}$, the number of paths from $(0, 0)$ to (n, m) is*

$$\binom{n}{\frac{n+m}{2}}.$$

Proof. Denote u as the number of up steps and d as the number of down steps. The total number of steps is

$$u + d = n$$

and the net number of up steps is

$$u - d = m.$$

Solving for u gives

$$u = \frac{n + m}{2},$$

and the lemma follows, since paths are determined by specifying which of the n steps are up steps. □

1.1 Proof of The Ballot Theorem

Proof of 1.1. The Reflection Principle shows that for $k, n, m > 0$, the number of lattice paths from $(0, k)$ to (n, m) which touch the t-axis (horizontal axis) is equal to the number of paths from $(0, -k)$ to (n, m).

Suppose Candidate A gets m votes and Candidate B gets n votes. We want to see how many ways we can count the votes such that A is always in the lead. Votes will be shown with a lattice path from $(0, 0)$ to $(m + n, m - n)$. Then the number of lattice paths from $(0, 0)$ to $(m + n, m - n)$ is

$$\binom{m+n}{n}.$$

Now, consider the number of ways with Candidate A always in the lead. Candidate A must always get the first vote. Then we need to find the number of ways of counting the votes where A is always in the lead while getting the first vote and the number of ways of counting the votes where A gets the first vote, but B is tied or ahead of A at least one time. In other words, we need to determine the number of lattice paths from $(1, 1)$ to $(m+n, m-n)$ and subtract from it the number of lattice paths from $(1, 1)$ to $(m + n, m - n)$ which touch the horizontal t-axis.

We can change one of the starting points $(1, 1)$ to $(1, -1)$ since, according to the Reflection Principle, the number of lattice paths will remain the same.

Then we can compare the number of lattice paths from $(1, 1)$ to $(m + n, m - n)$ and subtract from it the number of lattice paths from $(1, -1)$ to $(m + n, m - n)$. Now, suppose we change the starting position to the origin, then we will be subtracting the number of lattice paths from $(0, 0)$ to $(m + n - 1, m - n - 1)$ from the number of lattice paths from $(0, 0)$ to $(m + n - 1, m - n + 1)$.

Then we have

$$\begin{aligned}
\#\text{paths} &= \binom{m+n-1}{\frac{(m+n-1)+(m-n-1)}{2}} - \binom{m+n-1}{\frac{(m+n-1)+(m-n+1)}{2}} \\
&= \binom{m+n-1}{m-1} - \binom{m+n-1}{m} \\
&= \frac{(m+n-1)!}{(m-1)!n!} - \frac{(m+n-1)!}{m!(n-1)!} \\
&= \frac{(m-n)(m+n-1)!}{m!n!} \\
&= \frac{m-n}{m+n}\binom{m+n}{m}.
\end{aligned}$$

Therefore, the probability that Candidate A is always in the lead is

$$P(A \text{ is always in the lead}) = \frac{\frac{m-n}{m+n}\binom{m+n}{m}}{\binom{m+n}{m}} = \frac{m-n}{m+n}. \qquad \square$$

2 Simple Random Walk

Simple Symmetric Random Walk on $\mathbb{Z}$ is the process $S_0, S_1, \ldots$ defined by $S_0 = 0$ and $S_n = \sum_{k=1}^{n} X_k$ where $X_1, X_2, \ldots$ are i.i.d. with $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$. The trajectory of a simple random walk can be pictured as a polygonal path having height S_n at time n. Note that

$$P(\text{any particular possible path of length } n) = \frac{1}{2^n}.$$

We can see that $P(S_{2n+1} = 0) = 0$ because we need an even number of steps to return back to the t-axis. The time $2n$ return probability is

$$\begin{aligned} P(S_{2n} = 0) &= \frac{\text{the number of paths from } (0,0) \text{ to } (2n,0)}{2^{2n}} \\ &= 4^{-n}\binom{2n}{n}, \end{aligned}$$

since out of the $2n$ total steps, we need to chose which n are up steps.

Now, we want to show that the probability that the lattice path touches the t-axis on the $2n^{\text{th}}$ step is the same as the probability that the path does not touch the t-axis before or at the $2n^{\text{th}}$ step.

Theorem 2.1. *$P(S_{2n} = 0) = P(S_1, S_2, \ldots, S_{2n} \neq 0)$*

Proof. First, we know from above that

$$P(S_{2n} = 0) = 4^{-n}\binom{2n}{n}.$$

Next observe that

$$\begin{aligned} P(S_1, S_2, \ldots, S_{2n} \neq 0) &= P(S_1, \ldots, S_{2n} > 0) + P(S_1, \ldots, S_{2n} < 0) \\ &= 2P(S_1, \ldots, S_{2n} > 0), \end{aligned}$$

and

$$\begin{aligned} P(S_1, \ldots, S_{2n} > 0) &= P\Big(\bigcup_{r=1}^{n} \{S_1, \ldots, S_{2n} > 0, S_{2n} = 2r\}\Big) \\ &= \sum_{r=1}^{n} P(S_1, \ldots, S_{2n} > 0, S_{2n} = 2r). \end{aligned}$$

Since

$$\begin{aligned}
&P(S_1 = 1, S_2, \ldots, S_{2n-1} > 0, S_{2n} = 2r) \\
&\quad = \frac{1}{2}\frac{\text{number of paths from } (1,1) \text{ to } (2n,2r) \text{ which never touch } t\text{-axis}}{2^{2n-1}} \\
&\quad = \frac{\text{number of paths from } (1,1) \text{ to } (2n,2r) - \text{ number of paths from } (0,0) \text{ to } (2n-1,2r+1)}{2^{2n}} \\
&\quad = \frac{\binom{2n-1}{n+r-1} - \binom{2n-1}{n+r}}{2^{2n}},
\end{aligned}$$

we have

$$\begin{aligned}
P(S_1, \ldots, S_{2n} > 0) &= \sum_{r=1}^{n} P(S_1 = 1, S_2, \ldots, S_{2n-1} > 0, S_{2n} = 2r) \\
&= \frac{1}{2^{2n}} \sum_{r=1}^{n} \left[\binom{2n-1}{n+r-1} - \binom{2n-1}{n+r} \right] \\
&= \frac{1}{4^n} \left[\binom{2n-1}{n} - \binom{2n-1}{2n} \right] = \frac{1}{4^n} \binom{2n-1}{n},
\end{aligned}$$

so

$$\begin{aligned}
P(S_1, \ldots, S_{2n} \neq 0) &= \frac{1}{4^n} 2 \binom{2n-1}{n} \\
&= \frac{1}{4^n} 2 \frac{(2n-1)!}{n!(n-1)!} \frac{n}{n} \\
&= \frac{1}{4^n} \frac{(2n)!}{n!n!} = 4^{-n} \binom{2n}{n}
\end{aligned}$$

□

3 Recurrence

One can also consider simple random walks with asymmetric increment distributions (so that $P(X = 1) \neq P(X = -1)$), as well as simple random walks in higher dimensions. On $\mathbb{Z}^d$, simple symmetric random walk is the process defined by $S_n = \sum_{k=1}^{n} X_k$ where the X_k's are independent and equally likely to be any of the $2d$ vectors

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix} \in \mathbb{Z}^d.$$

There are also random walks which are not simple, meaning that the walker is not constrained to move only to neighboring sites.

Definition 3.1. A random walk S_n is *recurrent* if $P(S_n = 0 \text{ i.o.}) = 1$. That is, S_n is recurrent iff the lattice path touches the t-axis infinitely often. Otherwise, the random walk is *transient.*

To state our next result, we define $\tau_0 = 0$ and $\tau_n = \inf\{k > \tau_{n-1} : S_k = 0\}$ for $n \geq 1$. In other words, τ_n is the time of the n^{th} return to 0.

Lemma 3.1. *For any random walk, the following are equivalent:*

1. $P(\tau_1 < \infty) = 1$
2. $P(S_n = 0 \text{ } i.o.) = 1$
3. $\displaystyle\sum_{n=1}^{\infty} P(S_n = 0) = \infty$

Proof. First note that $P(\tau_n < \infty) = P(\tau_1 < \infty)^n$ for all $n \in \mathbb{N}$. Indeed this holds trivially when $n = 1$, and if $P(\tau_n < \infty) = P(\tau_1 < \infty)^n$, then

$$\begin{aligned} P(\tau_{n+1} < \infty) &= P(\tau_{n+1} < \infty | \tau_n < \infty) P(\tau_n < \infty) \\ &= P(\tau_1 < \infty) P(\tau_n < \infty) = P(\tau_1 < \infty)^{n+1}. \end{aligned}$$

If $P(\tau_1 < \infty) = 1$, then $P(\tau_n < \infty) = 1^n = 1$ for all n, so $P(S_n = 0 \text{ i.o.}) = 1$. Using the converse of Borel-Cantelli 1, we see that if $P(S_n = 0 \text{ i.o.}) = 1$, then $\sum_{n=1}^{\infty} P(S_n = 0) = \infty$. Finally, $\sum_{n=1}^{\infty} P(S_n = 0) = \infty$ implies $P(\tau_1 < \infty) = 1$ since considering N as the number of times the simple random walk touches the t-axis gives

$$\begin{aligned} N &= \sum_{k=1}^{\infty} 1\{S_k = 0\} \\ &= \sum_{k=1}^{\infty} 1\{\tau_k < \infty\}, \end{aligned}$$

so, taking $p = P(\tau_1 < \infty)$,

$$\begin{aligned} \sum_{k=1}^{\infty} P(S_k = 0) = E[N] &= \sum_{k=1}^{\infty} P(\tau_k < \infty) \\ &= \sum_{k=1}^{\infty} p^k = \frac{p}{1-p} \text{ if } p < 1. \end{aligned}$$

Therefore, all the above statements are equivalent and can be derived from each other. $\square$

Theorem 3.2. *Simple random walk is recurrent in dimensions one and two.*

Proof. When $d = 1$, $P(S_{2n-1} = 0) = 0$ and Stirling's formula gives

$$\begin{aligned}
P(S_{2n} = 0) &= \frac{1}{2^{2n}}\binom{2n}{n} = \frac{1}{2^{2n}}\frac{(2n)!}{(n!)^2} \\
&\approx \frac{1}{2^{2n}}\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)^2} \\
&= \frac{1}{4^n}\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}} \\
&= \frac{2^{2n}}{4^n}\frac{\left(\frac{n}{e}\right)^{2n}\sqrt{4\pi n}}{\left(\frac{n}{e}\right)^{2n}2\pi n} = \frac{1}{\sqrt{\pi n}}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{\pi n}}$ diverges, it follows from the limit comparison test that

$$\sum_{n=1}^{\infty} P(S_n = 0) = \sum_{n=1}^{\infty} P(S_{2n} = 0) = \infty,$$

so $P(S_n = 0 \text{ i.o.}) = 1$ and S_n is recurrent in one dimension.

Similarly, when $d = 2$, $P(S_{2n-1} = 0) = 0$ and

$$\begin{aligned}
P(S_{2n} = 0) &= \frac{1}{4^{2n}}\sum_{k=0}^{n}\binom{2n}{2k}\binom{2n-2k}{n-k}\binom{2k}{k} \\
&= \frac{1}{4^{2n}}\sum_{k=0}^{n}\frac{(2n)!}{(2k)!(2n-2k)!}\frac{(2n-2k)!}{(n-k)!(n-k)!}\frac{(2k)!}{k!k!} \\
&= \frac{1}{4^{2n}}\sum_{k=0}^{n}\frac{2n!}{k!^2(n-k)!^2} \\
&= \frac{1}{4^{2n}}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2 \\
&= \left(\frac{1}{2^{2n}}\right)^2\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} \\
&= \left[\frac{1}{2^{2n}}\binom{2n}{n}\right]^2 = \left[\frac{1}{2^{2n}}\left(\frac{2n!}{n!n!}\right)\right]^2 \\
&\approx \left[\frac{1}{2^{2n}}\left(\frac{2\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}\right)\right]^2 \\
&= \frac{1}{2^{4n}}\left(\frac{4\left(2\pi n\right)\left(\frac{n}{e}\right)^n}{2\pi n\left(\frac{n}{e}\right)^n 2\pi n\left(\frac{n}{e}\right)^n}\right) = \frac{1}{\pi n}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\frac{1}{\pi n}$ diverges, we see that S_n is recurrent in two dimensions as well. □

It follows from the previous result that when $d = 1$ every site in $\mathbb{Z}$ is visited infinitely often with probability one. However, one can show that the expected time to travel between any two sites (or return to the present site) is infinite!

Theorem 3.3. *Simple random walk is transient in three or more dimensions.*

Proof. When $d = 3$,

$$P(S_{2n} = 0) = 6^{-2n} \sum_{\substack{n_1,n_2,n_3 \geq 0: \\ n_1+n_2+n_3=n}} \frac{(2n)!}{(n_1!n_2!n_3!)^2}$$
$$= 2^{-2n}\binom{2n}{n} \sum_{\substack{n_1,n_2,n_3 \geq 0: \\ n_1+n_2+n_3=n}} \left(3^{-n}\frac{n!}{n_1!n_2!n_3!}\right)^2.$$

Now $3^{-n}\frac{n!}{n_1!n_2!n_3!} \geq 0$ for each choice of n_1, n_2, n_3, n, and the multinomial theorem gives

$$\sum_{\substack{n_1,n_2,n_3 \geq 0: \\ n_1+n_2+n_3=n}} 3^{-n}\frac{n!}{n_1!n_2!n_3!} = \sum_{\substack{n_1,n_2,n_3 \geq 0: \\ n_1+n_2+n_3=n}} \binom{n}{n_1, n_2, n_3} \left(\frac{1}{3}\right)^{n_1}\left(\frac{1}{3}\right)^{n_2}\left(\frac{1}{3}\right)^{n_3} = \left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) = 1,$$

so

$$\sum_{\substack{n_1,n_2,n_3 \geq 0: \\ n_1+n_2+n_3=n}} \left(3^{-n}\frac{n!}{n_1!n_2!n_3!}\right)^2 \leq \left(\max_{\substack{0 \leq n_1 \leq n_2 \leq n_3: \\ n_1+n_2+n_3=n}} 3^{-n}\frac{n!}{n_1!n_2!n_3!}\right) \sum_{\substack{n_1,n_2,n_3 \geq 0: \\ n_1+n_2+n_3=n}} 3^{-n}\frac{n!}{n_1!n_2!n_3!}$$
$$= 3^{-n} \max_{\substack{0 \leq n_1 \leq n_2 \leq n_3: \\ n_1+n_2+n_3=n}} \frac{n!}{n_1!n_2!n_3!}$$

The latter quantity is maximized when $n_1!n_2!n_3!$ is minimized. This happens when n_1, n_2, n_3 are as close as possible: If $n_i < n_j - 1$ for $i < j$, then $n_i!n_j! > \frac{n_i+1}{n_j} n_i!n_j! = (n_i+1)!(n_j-1)!$.

It follows that

$$\max_{\substack{0 \leq n_1 \leq n_2 \leq n_3: \\ n_1+n_2+n_3=n}} \frac{n!}{n_1!n_2!n_3!} \approx \frac{n!}{\left(\left[\frac{n}{3}\right]!\right)^3} \approx \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\left(\sqrt{\frac{2\pi n}{3}}\left(\frac{n}{3e}\right)^{\frac{n}{e}}\right)^3} = \frac{3^{\frac{3}{2}}\left(\frac{n}{e}\right)^n}{2\pi n\left(\frac{n}{3e}\right)^n} \leq \frac{3^n}{n}.$$

Putting all this together and recalling that $\frac{1}{2^{2n}}\binom{2n}{n} \approx \frac{1}{\sqrt{\pi n}}$ shows that

$$P(S_{2n} = 0) = 2^{-2n}\binom{2n}{n} \sum_{\substack{n_1,n_2,n_3 \geq 0: \\ n_1+n_2+n_3=n}} \left(3^{-n}\frac{n!}{n_1!n_2!n_3!}\right)^2 \leq 2^{-2n}\binom{2n}{n}\frac{1}{n} \approx \frac{c}{n^{\frac{3}{2}}},$$

where c is a constant.

Hence $\sum_{n=1}^{\infty} P(S_n = 0) < \infty$ and we conclude that Simple Random Walk is transient in 3-dimensions.

Transience in higher dimensions follows by letting $T_n = (S_n^1, S_n^2, S_n^3)$ be the projection onto the first three coordinates and letting $N(n) = \inf\{m > N(n-1) : T_m \neq T_{N(n-1)}\}$ to be the n^{th} time that the random walker moves in any of the first three coordinates (with the convention that $N(0) = 0$). Then $T_{N(n)}$ is a simple random walk in three dimensions and the probability that $T_{N(n)} = 0$ infinitely often is 0. Since the first three coordinates of S_n are constant between $N(n)$ and $N(n+1)$ and $N(n+1) - N(n)$ is almost surely finite, this implies that S_n is transient. □

4 Arcsine Laws

In this section, we focus on simple random walk on $\mathbb{Z}$. Define

$$L_n = \max\{0 \le k \le n : S_k = 0\}$$

to be the time of the last visit to zero by time n.

Lemma 4.1. *Let $u_{2m} = P(S_{2m} = 0)$. Then $P(L_{2n} = 2k) = u_{2k}u_{2n-2k}$ for $k = 0, 1, \ldots n$.*

Proof.

$$\begin{aligned} P(L_{2n} = 2k) &= P(S_{2k} = 0, S_{2k+1} \neq 0, \ldots, S_{2n} \neq 0) \\ &= P(S_{2k} = 0, X_{2k+1} \neq 0, \ldots, X_{2k+1} + \ldots + X_{2n} \neq 0) \\ &= P(S_{2k} = 0)P(X_{2k+1} \neq 0, \ldots, X_{2k+1} + \ldots + X_{2n} \neq 0) \\ &= P(S_{2k} = 0)P(S_1 \neq 0, \ldots, S_{2n-2k} \neq 0) = u_{2k}u_{2n-2k}. \end{aligned}$$

□

The preceding observation allows us to prove the *second arcsine law.*

Theorem 4.2. *For $0 < a < b < 1$,*

$$P\left(a \le \frac{L_{2n}}{2n} \le b\right) \to \int_a^b \frac{1}{\pi\sqrt{x(1-x)}}dx.$$

Proof. We first note that

$$nP(L_{2n} = 2k) = nu_{2k}u_{2(n-k)} \approx \frac{n}{\sqrt{\pi k}\sqrt{\pi(n-k)}} = \frac{1}{\pi}\frac{1}{\sqrt{\frac{k}{n}(1-\frac{k}{n})}},$$

so if $\frac{k}{n} \to x$, then

$$nP(L_{2n} = 2k) = \left(\frac{nP(L_{2n} = 2k)}{\frac{1}{\pi\sqrt{\frac{k}{n}(1-\frac{k}{n})}}}\frac{1}{\pi\sqrt{\frac{k}{n}(1-\frac{k}{n})}}\right) \to \frac{1}{\pi\sqrt{x(1-x)}}.$$

Now, define a_n and b_n, so that $2na_n$ is the smallest even integer greater than or equal to $2na$ and $2nb_n$ is the largest even integer less than or equal to $2nb$.

Setting $f_n(x) = nP(L_{2n} = 2k)$ for $\frac{k}{n} \leq x < \frac{(k+1)}{n}$, we have

$$P\Big(a \leq \frac{L_{2n}}{2n} \leq b\Big) = P(2na_n \leq L_{2n} \leq 2nb_n) = \sum_{k=na_n}^{nb_n} nP(L_{2n} = 2k)\frac{1}{n} = \int_{a_n}^{b_n+\frac{1}{n}} f_n(x)dx.$$

Using ideas from real analysis, one can show that this implies

$$P\Big(a \leq \frac{L_{2n}}{2n} \leq b\Big) = \int_{a_n}^{b_n+\frac{1}{n}} f_n(x)dx \to \int_a^b f(x)dx. \qquad \square$$

5 Appendix

Proposition 1 (Stirling's Formula)**.**

$$n! \approx \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n$$

where $a_n \approx b_n$ *means* $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

Proposition 2 (Continuity From Below)**.** *If* $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, *then*

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \lim_{n\to\infty} P(A_n)$$

Proof. Set $B_1 = A_1, B_2 = A_2 \backslash A_1, \ldots, B_k = A_k \backslash \bigcup_{i=1}^{k-1} A_i, \ldots$. Then $B_1, B_2, \ldots$ are disjoint, with

$$\bigcup_{j=1}^{k} B_k = A_k \text{ and } \bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} A_j.$$

Thus

$$\begin{aligned} P\Big(\bigcup_{j=1}^{\infty} A_j) &= P\Big(\bigcup_{j=1}^{\infty} B_j\Big) \\ &= \sum_{j=1}^{\infty} P(B_j) \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} P(B_j) \\ &= \lim_{n\to\infty} P\Big(\bigcup_{j=1}^{n} B_j\Big) \\ &= \lim_{n\to\infty} P(A_n). \end{aligned} \qquad \square$$

Proposition 3 (Continuity Above)**.** *If* $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$, *then*

$$P\Big(\bigcap_{i=1}^{\infty} A_i\Big) = \lim_{n\to\infty} P(A_n).$$

Proof. If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$, then $A_1^C \subseteq A_2^C \subseteq \ldots$, so

$$\begin{aligned} P\Big(\bigcap_{i=1}^{\infty} A_i\Big) &= 1 - P\Big(\Big(\bigcap_{i=1}^{\infty} A_i\Big)^C\Big) \\ &= 1 - P\Big(\bigcup_{i=1}^{\infty} A_i^C\Big) \\ &= 1 - \lim_{n\to\infty} P(A_n^C) \\ &= 1 - \lim_{n\to\infty} (1 - P(A_n)) \\ &= \lim_{n\to\infty} P(A_n) \end{aligned}$$

□

Proposition 4 (Borel-Cantelli I)**.** *If* $A_1, A_2, \ldots$ *are events with* $\sum_{n=1}^{\infty} P(A_n) < \infty$, *then*

$$P(A_n \ i.o.) := P\Big(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\Big) = 0.$$

Proof. If $B_n = \bigcup_{m=n}^{\infty} A_m$, then $B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$, so

$$\begin{aligned} P(A_n \ i.o.) &= P\Big(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\Big) \\ &= P\Big(\bigcap_{n=1}^{\infty} B_n\Big) \\ &= \lim_{n\to\infty} P(B_n). \end{aligned}$$

The result follows since $\sum_{n=1}^{\infty} P(A_n) < \infty$ implies

$$P(B_n) = P\Big(\bigcup_{m=n}^{\infty} A_m\Big) \le \sum_{m=n}^{\infty} P(A_m) \to 0$$

as $n \to \infty$. □