$$C = k_1\mu_1 + k_2\mu_2 + \cdots + k_t\mu_t \qquad \underset{\sim}{K} = \begin{pmatrix} k_1 \\ \vdots \\ k_t \end{pmatrix} \quad \underset{\sim}{\bar{Y}} = \begin{pmatrix} \bar{Y}_{1\bullet} \\ \vdots \\ \bar{Y}_{t\bullet} \end{pmatrix}$$

$$\sum_{i=1}^{t} k_i = 0$$

$$\hat{C} = k_1\bar{Y}_{1\bullet} + k_2\bar{Y}_{2\bullet} + \cdots + k_t\bar{Y}_{t\bullet} \qquad \hat{C} = (k_1 \cdots k_t)\begin{pmatrix} \bar{Y}_{1\bullet} \\ \vdots \\ \bar{Y}_{t\bullet} \end{pmatrix} = \sum_{i=1}^{t} k_i\bar{Y}_{i\bullet} = \underset{\sim}{K}\bullet\underset{\sim}{\bar{Y}} = \underset{\sim}{K}^T\underset{\sim}{\bar{Y}}$$

$$C_1 = \mu_1 - \mu_2$$
$$C_2 = \mu_1 - \mu_3$$
$$C_3 = \mu_2 - \mu_3$$
$$C_4 = \frac{\mu_1 + \mu_2}{2} - \mu_3$$

$$C_2 - \frac{1}{2}C_1 = (\mu_1 - \mu_3) - \frac{1}{2}(\mu_1 - \mu_2) = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 - \mu_3 = C_4$$

A set of contrasts are linearly independent if they are orthogonal.

Consider two vectors : $\underset{\sim}{A}$ & $\underset{\sim}{B}$

$$\underset{\sim}{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \underset{\sim}{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\underset{\sim}{A}\bullet\underset{\sim}{B} = \underset{\sim}{A}^T\underset{\sim}{B} = a_1b_1 + a_2b_2 + a_3b_3 = \|\underset{\sim}{A}\|\|\underset{\sim}{B}\|\cos\theta$$

- Two contrasts $\begin{cases} C = c_1\mu_1 + c_2\mu_2 + \cdots + c_t\mu_t \\ D = d_1\mu_1 + d_2\mu_2 + \cdots + d_t\mu_t \end{cases}$ are orthogonal if $\sum_{i=1}^{t} c_i d_i = 0$

Model: $Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$, $\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_e^2)$

→ $\hat{\mu} = \bar{Y}_{\bullet\bullet}$ & $\hat{\alpha}_i = \bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet}$ are Least Squares Estimators

- If $\hat{\mu}_i$ is the LSE of μ_i then $c_1\hat{\mu}_1 + c_2\hat{\mu}_2 + \cdots + c_n\hat{\mu}_n$ is the LSE of $c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n$

Density	**Mean_Yield**
10	12
20	16
30	19
40	18
50	17

```
Class     Levels    Values

DENSITY        5    10 20 30 40 50
```

```
Dependent Variable: YIELD
                                      Sum of          Mean
Source                     DF        Squares        Square  F Value    Pr > F
Model                       4     87.6000000    21.9000000    29.28    0.0001
Error                      10      7.4800000     0.7480000
Corrected Total            14     95.0800000

                     R-Square           C.V.      Root MSE         YIELD Mean
                     0.921329       5.273597       0.86487            16.4000
```

$$\hat{Y}_{ij} = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 X^2 + \cdots + \hat{\beta}_k X^k$$
$$= \hat{\beta}_0^* + \hat{\beta}_1^* P_{1i} + \hat{\beta}_2^* P_{2i} + \cdots + \hat{\beta}_k^* P_{ki}$$

$$\hat{\beta}_0^* = \bar{Y}_{\bullet\bullet}$$

$$\hat{\beta}_1^* = \frac{\hat{C}_{lin}}{\sum_i c_{i,lin}^2} = \frac{\sum_{i=1}^{t} P_{1i}\bar{Y}_{i\bullet}}{\sum_i P_{1\bullet}^2}$$

$$\hat{\beta}_2^* = \frac{\hat{C}_{quad}}{\sum_i c_{i,quad}^2} = \frac{\sum_{i=1}^{t} P_{2i}\bar{Y}_{i\bullet}}{\sum_i P_{2\bullet}^2}$$

$\cdots$

Table XI:

$k = 1: \; (P_{11}, P_{12}, \ldots, P_{15}) = (-2, -1, \; 0, \; 1, \; 2)$

$k = 2: \; (P_{21}, P_{22}, \ldots, P_{25}) = (\; 2, -1, -2, -1, \; 2)$

$\cdots$

Density	P1	P2	…	Y_hat(degree 2 poly)
10	-2	2		12.0
20	-1	-1		16.2
30	0	-2		18.4
40	1	-1		18.6
50	2	2		16.8

t=3 Treatments, r=2 replicates

Cell Means Model:

$$
\begin{aligned}
Y_{11} &= \mu_1 + \varepsilon_{11}\\
Y_{12} &= \mu_1 + \varepsilon_{12}\\
Y_{21} &= \mu_2 + \varepsilon_{21}\\
Y_{22} &= \mu_2 + \varepsilon_{22}\\
Y_{31} &= \mu_3 + \varepsilon_{31}\\
Y_{32} &= \mu_3 + \varepsilon_{32}
\end{aligned}
$$

$$
\begin{bmatrix} Y_{11}\\ Y_{12}\\ Y_{21}\\ Y_{22}\\ Y_{31}\\ Y_{32} \end{bmatrix} = \begin{bmatrix} 1&0&0\\ 1&0&0\\ 0&1&0\\ 0&1&0\\ 0&0&1\\ 0&0&1 \end{bmatrix} \begin{bmatrix} \beta_1\\ \beta_2\\ \beta_3 \end{bmatrix} + \begin{bmatrix} \varepsilon_{11}\\ \varepsilon_{12}\\ \varepsilon_{21}\\ \varepsilon_{22}\\ \varepsilon_{31}\\ \varepsilon_{32} \end{bmatrix}
$$

$$
\underset{\sim}{Y} = \underset{\sim}{X}\,\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}
$$

Treatment Effects Model:

$$
\begin{aligned}
Y_{11} &= \mu + \alpha_1 + \varepsilon_{11}\\
Y_{12} &= \mu + \alpha_1 + \varepsilon_{12}\\
Y_{21} &= \mu + \alpha_2 + \varepsilon_{21}\\
Y_{22} &= \mu + \alpha_2 + \varepsilon_{22}\\
Y_{31} &= \mu + \alpha_3 + \varepsilon_{31}\\
Y_{32} &= \mu + \alpha_3 + \varepsilon_{32}
\end{aligned}
$$

$$
\begin{bmatrix} Y_{11}\\ Y_{12}\\ Y_{21}\\ Y_{22}\\ Y_{31}\\ Y_{32} \end{bmatrix} = \begin{bmatrix} 1&1&0\\ 1&1&0\\ 1&0&1\\ 1&0&1\\ 1&0&0\\ 1&0&0 \end{bmatrix} \begin{bmatrix} \beta_0\\ \beta_1\\ \beta_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_{11}\\ \varepsilon_{12}\\ \varepsilon_{21}\\ \varepsilon_{22}\\ \varepsilon_{31}\\ \varepsilon_{32} \end{bmatrix}
$$

$$
\underset{\sim}{Y} = \underset{\sim}{X}\,\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}
$$

Bonferroni's 100(1-α)% SCI (for k planned contrasts)

$$
\sum_{i=1}^{t} c_i \bar{Y}_{i\bullet} \pm t_{\frac{\alpha_{EW}/2}{k},\,N-t} \sqrt{s^2 \sum_{i=1}^{t} c_i^2 / r_i} \quad \text{(Table 2)}
$$

$$
\sum_{i=1}^{t} c_i \bar{Y}_{i\bullet} \pm t_{\alpha_{EW}/2,\,k,\,N-t} \sqrt{s^2 \sum_{i=1}^{t} c_i^2 / r_i} \quad \text{(Table 5)}
$$

Table 2 vs. Table 5 (e.g., k=5 contrasts, α_{EW} =0.05, df=10)

$$
t_{\frac{\alpha_{EW}/2}{k},\,N-t} = t_{\frac{0.05/2}{5},\,10} = t_{0.005,\,10} = 3.169 \quad \text{(Table 2)}
$$

$$
t_{\alpha_{EW}/2,\,k,\,N-t} = t_{0.05/2,\,5,\,10} = t_{0.025,\,5,\,10} = 3.17 \quad \text{(Table 5)}
$$

Rejection Region (Bonferroni correction)

$$
|t_0| > t_{\frac{\alpha_{EW}/2}{k},\,N-t} \quad \text{(Table 2)}
$$

$$
|t_0| > t_{\alpha_{EW}/2,\,k,\,N-t} \quad \text{(Table 5)}
$$

$$C = \sum_{i=1}^{t} c_i \mu_i \qquad \hat{C} = \sum_{i=1}^{t} c_i \bar{Y}_{i\bullet} \qquad t_0 = \frac{\hat{C}}{SE(\hat{C})} = \frac{\sum_{i=1}^{t} c_i \bar{Y}_{i\bullet}}{\sqrt{s^2 \sum_{i=1}^{t} c_i^2 / r_i}}$$

100(1-α)% SCI for several *C*'s

$\hat{C} \qquad \pm w\, SE(\hat{C})$

$\hat{C} \qquad \pm MSD$

$\sum_{i=1}^{t} c_i \bar{Y}_{i\bullet} \pm w \sqrt{s^2 \sum_{i=1}^{t} c_i^2 / r_i}$

Bonferroni's Method

* k planned contrasts $\qquad \hat{C} = \sum_{i=1}^{t} c_i \bar{Y}_{i\bullet}$
* $MSD = t_{\frac{\alpha_{EW}/2}{k}, N-t} \sqrt{s^2 \sum_{i=1}^{t} c_i^2 / r_i}$

Scheffé's Method

* all possible contrasts $\qquad \hat{C} = \sum_{i=1}^{t} c_i \bar{Y}_{i\bullet}$
* $MSD = \sqrt{(t-1) F_{\alpha, t-1, N-t}} \sqrt{s^2 \sum_{i=1}^{t} c_i^2 / r_i}$

Dunnett's Method

* comparing k treatments with a control (r_i=r) $\qquad \hat{C} = \bar{Y}_{i\bullet} - \bar{Y}_{Control}$
* $MSD = d_{\alpha,k,N-t} \sqrt{2s^2 / r}$
* Reject Ho if $\left|\bar{Y}_{i\bullet} - \bar{Y}_{Control}\right| > MSD$
* Table VI for 1-sided and 2-sided critical values for $d_{\alpha,k,N-t}$

Tukey's HSD Method

* all pairwise comparisons (r_i=r) $\qquad \hat{C} = \bar{Y}_{i\bullet} - \bar{Y}_{j\bullet}$
* $MSD = q_{\alpha,k,N-t} \sqrt{s^2 / r}$
* Reject Ho if $\left|\bar{Y}_{i\bullet} - \bar{Y}_{j\bullet}\right| > MSD$
* Unequal replications → Tukey-Kramer Method with $MSD = q_{\alpha,k,N-t} \sqrt{\frac{s^2}{2}\left(\frac{1}{r_i} + \frac{1}{r_j}\right)}$

Fisher's LSD Method (Protected T-tests)

* If the omnibus F-test from the ANOVA is not significant then STOP
  Otherwise, test each pair with an ordinary T-test
* Reject Ho if $\dfrac{\bar{Y}_{i\bullet} - \bar{Y}_{j\bullet}}{\sqrt{s^2\left(\frac{1}{r_i} + \frac{1}{r_j}\right)}} > t_{\alpha/2, N-t}$
* Controls α_{EW} in the weak sense through the overall F-test

Student Newamn Keuls (SNK) Method

* One of several multiple range tests
* "step-down procedure" since you are looking at different ranges of means

Ryan Einot Gabriel Welsch (REGWQ) Method

* Another multiple range test

Multiple Comparisons with Best (MCB) Method

* Similar to Dunnett's procedure
  uses 1-sided critical values for $d_{\alpha,k,N-t}$
* less commonly used in biostatistics, more common in engineering applications