$Y_i \overset{iid}{\sim} N(\mu,\sigma^2)$ → $\hat{\mu} = \bar{Y}$ & $\hat{\sigma}^2 = \sum_{i=1}^{n} \frac{(Y_i - \bar{Y})^2}{n-1}$ are *unbiased*

$\hat{\sigma}^2$ *unbiased* for σ^2 → $\boxed{E\left[\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2\right] = (n-1)\sigma^2}$

$z_i = \frac{Y_i - \mu}{\sigma} \overset{iid}{\sim} N(0,1)$ → $z_i^2 = \left(\frac{Y_i - \mu}{\sigma}\right)^2 \sim \chi_1^2$ $\quad z_1^2 + z_2^2 \sim \chi_2^2$

$\sum_{i=1}^{n} z_i^2 \sim \chi_n^2$ → $\sum_{i=1}^{n} \frac{(Y_i - \mu)^2}{\sigma^2} \sim \chi_n^2$

Estimating μ with $\hat{\mu} = \bar{Y}$ → $\sum_{i=1}^{n} \frac{(Y_i - \bar{Y})^2}{\sigma^2} \sim \chi_{n-1}^2$

→ $\frac{(n-1)\hat{\sigma}^2}{\sigma^2} \sim \chi_{n-1}^2$

$H_o : \sigma_1^2 = \sigma_2^2$ $\quad \hat{\sigma}_1^2 = \frac{\sigma_1^2 \chi_{m-1}^2}{m-1}$ $\quad \hat{\sigma}_2^2 = \frac{\sigma_2^2 \chi_{n-1}^2}{n-1}$

$$\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2} = \frac{\sigma_1^2 \chi_{m-1}^2 / m-1}{\sigma_2^2 \chi_{n-1}^2 / n-1} \overset{*}{=} \frac{\chi_{m-1}^2 / m-1}{\chi_{n-1}^2 / n-1} = F_{m-1,n-1} \quad *\text{under } H_o$$

Central vs. Non-central Chi-Square Random Variables

$z_i \sim N(0,1)$ → $\sum_{i=1}^{n} z_i^2 \sim \chi_n^2$ (Central)

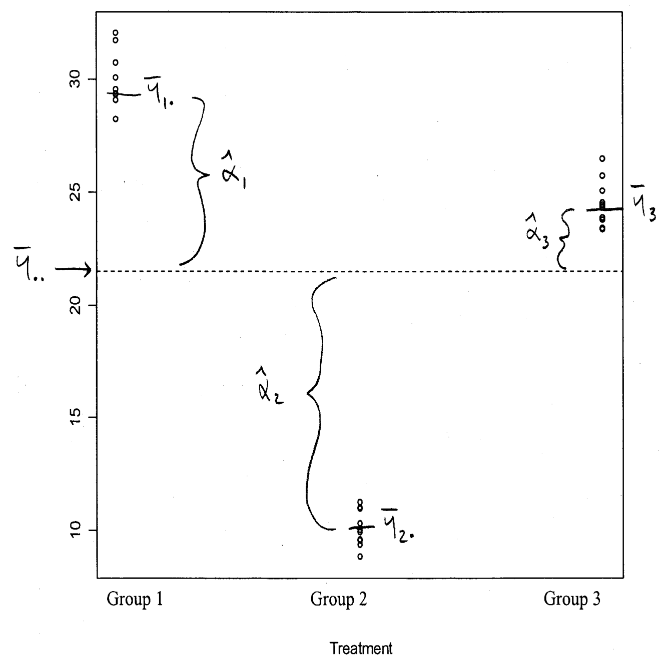
$w_i \sim N(\mu_i, 1)$ → $\sum_{i=1}^{n} w_i^2 \sim \chi_n^2$ (Non-Central with non-centrality parameter λ)

$$\lambda = \sqrt{\sum_{i=1}^{n} \mu_i^2}$$

Treatments: 1, 2, …, t

Y_{ij} : j^{th} observation of the i^{th} treatment group

$Group_1$	$Group_2$ …		$Group_t$	
Y_{11}	Y_{21}	…	Y_{t1}	
Y_{12}	Y_{22}	…	Y_{t2}	
…	…	…	…	
Y_{1n}	Y_{2n}	…	Y_{tn}	
$Y_{1\cdot}$	$Y_{2\cdot}$	...	$Y_{t\cdot}$	Column Sum
$\bar{Y}_{1\cdot}$	$\bar{Y}_{2\cdot}$	...	$\bar{Y}_{t\cdot}$	Column Average



'Dot-bar' Notation: $Y_{2\bullet} = \sum_{j=1}^{r} Y_{2j}$ $\qquad \bar{Y}_{2\bullet} = \frac{\sum_{j=1}^{r} Y_{2j}}{r}$

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t treatments $i = 1, 2, \ldots, t$
r replicates $j = 1, 2, \ldots, r$

Cell Means Model

$$Y_{ij} = \mu_i + \varepsilon_{ij}$$

Treatment Effects Model

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$$

μ : constant $\qquad \hat{\mu} = \bar{Y}_{\bullet\bullet}$

α_i : constant for the i^th group $\qquad \hat{\alpha}_i = \bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet}$

$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_e^2)$ $\qquad \hat{\varepsilon}_{ij} = Y_{ij} - \bar{Y}_{i\bullet}$

Least Squares Estimates (LSE)

$$\varepsilon_{ij} = Y_{ij} - \mu - \alpha_i \qquad (ij^{th} \ residual)$$

$$SSE = S = \sum_{i=1}^{t} \sum_{j=1}^{r} \varepsilon_{ij}^2 = \sum_{i=1}^{t} \sum_{j=1}^{r} \left(Y_{ij} - \mu - \alpha_i \right)^2$$

$\hat{\mu}$ & $\hat{\alpha}_i$ minimize SSE

Sum of Squares Decomposition

$$Y_{ij} - \hat{\mu} = \hat{\alpha}_i + \hat{\varepsilon}_{ij}$$

$$Y_{ij} - \bar{Y}_{\bullet\bullet} = (\bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet}) + (Y_{ij} - \bar{Y}_{i\bullet})$$

$$\sum_i \sum_j \left[Y_{ij} - \bar{Y}_{\bullet\bullet} \right]^2 = \sum_i \sum_j \left[(\bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet}) + (Y_{ij} - \bar{Y}_{i\bullet}) \right]^2$$

$$= \sum_i \sum_j (\bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet})^2 + 2 \sum_i \sum_j (\bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet})(Y_{ij} - \bar{Y}_{i\bullet}) + \sum_i \sum_j (Y_{ij} - \bar{Y}_{i\bullet})^2$$

$$= \sum_i \sum_j (\bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet})^2 + \sum_i \sum_j (Y_{ij} - \bar{Y}_{i\bullet})^2$$

$$SS(Total) = SS(TRT) + SS(Error)$$

$$SS_{Tot} = SST + SSE$$

Recall: $Y_{ij} \overset{iid}{\sim} N(\mu_i, \sigma^2) \rightarrow \boxed{E\left[\sum_{j=1}^{n}\left(Y_{ij} - \bar{Y}_{i\bullet}\right)^2\right] = (n-1)\sigma^2}$

	Source	d.f.	SS	MS	E(MS)
(Between)	Treatment (T)	*t-1*	$\sum_{i=1}^{t}\sum_{j=1}^{n_i}\left(\bar{Y}_{i\bullet} - \bar{Y}_{\bullet\bullet}\right)^2$	*SST / (t-1)*	$\sigma_e^2 + \frac{r\sum_{i=1}^{t}\alpha_i^2}{t-1}$
(Within)	Error (E)	*N-t*	$\sum_{i=1}^{t}\sum_{j=1}^{n_i}\left(Y_{ij} - \bar{Y}_{i\bullet}\right)^2$	*SSE / (N-t)*	σ_e^2
	Total	*N-1*	$\sum_{i=1}^{t}\sum_{j=1}^{n_i}\left(Y_{ij} - \bar{Y}_{\bullet\bullet}\right)^2$		

Model: $Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}\,, \quad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_e^2)$

Recall: $Y_{ij} \overset{iid}{\sim} N(\mu_i, \sigma^2) \rightarrow \boxed{E\left[\sum_{j=1}^{n}\left(Y_{ij} - \bar{Y}_{i\bullet}\right)^2\right] = (n-1)\sigma^2}$

$$\rightarrow \quad E\left[\sum_{j=1}^{r}\left(\varepsilon_{ij} - \bar{\varepsilon}_{i\bullet}\right)^2\right] = (r-1)\sigma_e^2$$

$$SSE = \sum_{i=1}^{t}\sum_{j=1}^{r}\left(Y_{ij} - \bar{Y}_{i\bullet}\right)^2 = \sum_{i=1}^{t}\sum_{j=1}^{r}\left[(\mu + \alpha_i + \varepsilon_{ij}) - (\mu + \alpha_i + \bar{\varepsilon}_{i\bullet})\right]^2$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{r}\left[\varepsilon_{ij} - \bar{\varepsilon}_{i\bullet}\right]^2$$

$$E(SSE) = \sum_{i=1}^{t} E\left[\sum_{j=1}^{r}\left(\varepsilon_{ij} - \bar{\varepsilon}_{i\bullet}\right)^2\right] = \sum_{i=1}^{t}(r-1)\sigma_e^2 = t(r-1)\sigma_e^2$$

$$MSE = \frac{SSE}{t(r-1)} \quad \rightarrow \quad E(MSE) = \sigma_e^2$$

Scheffe's Rule #2 (for computing E(MS) under the null hypothesis) in a Fixed Effects Model

$$E(MS\text{ X}) = \sigma_e^2 + c$$

where ***c*** is given by replacing, in MS***X***, each observation Y_{ij} by its expectation under H_o

e.g., $$MSE = \frac{SSE}{t(r-1)} = \frac{\sum_{i=1}^{t}\sum_{j=1}^{r}\left(Y_{ij} - \bar{Y}_{i\bullet}\right)^2}{t(r-1)}$$