Matrices to Simplify Regression Notation

SLR:

$$\begin{aligned} Y_1 &= \beta_0 + \beta_1 x_1 + \varepsilon_1 \\ Y_2 &= \beta_0 + \beta_1 x_2 + \varepsilon_2 \\ &\vdots \\ Y_n &= \beta_0 + \beta_1 x_n + \varepsilon_n \end{aligned}$$

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} 1 & X_1 \\ 1 & X_2 \\ 1 & \vdots \\ 1 & X_n \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$\underset{\sim}{Y} = \underset{\sim}{X} \; \underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$$

- Matrix Basics:
  - Matrix & Transpose: $\mathbf{A}_{r \times c} = (a_{ij})$ $\quad\quad$ $\mathbf{A}' = (a_{ji})$

$$\mathbf{A}_{2\times 3} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{B}_{2\times 3} = \begin{bmatrix} 4 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix} \qquad \mathbf{A}' = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}$$

  - Addition: $\mathbf{A}_{r \times c} + \mathbf{B}_{r \times c} = \mathbf{C}_{r \times c}$

$$\underset{2\times 3}{\mathbf{A} + \mathbf{B}} = \begin{bmatrix} 2+4 & 3+1 & 1+5 \\ 1+1 & 1+3 & 2+1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 6 \\ 2 & 4 & 3 \end{bmatrix}$$

  - Multiplication: $\mathbf{A}_{m \times n} \; \mathbf{B}_{n \times p} = \mathbf{C}_{m \times p}$ $\quad$ (for *conformable* matrices)

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} (2\times 1) + (1 \times 1) + (0 \times 3) = 3 & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 6 & 2 \end{bmatrix}$$

  - Identity: **A I = A** and **I A = A**

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - Inverse:

$$\mathrm{A} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \qquad \mathrm{A}^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

$$\mathbf{A} \; \mathbf{A}^{-1} = \mathbf{I} \qquad\qquad \mathbf{A}^{-1} \; \mathbf{A} = \mathbf{I}$$

$$\begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \mathrm{A}^{-1} = \frac{1}{\det \mathrm{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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SLR:

$$\begin{aligned} Y_1 &= \beta_0 + \beta_1 x_1 + \varepsilon_1 \\ Y_2 &= \beta_0 + \beta_1 x_2 + \varepsilon_2 \\ &\vdots \\ Y_n &= \beta_0 + \beta_1 x_n + \varepsilon_n \end{aligned} \qquad \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} 1 & X_1 \\ 1 & X_2 \\ 1 & \vdots \\ 1 & X_n \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$\underset{\sim}{Y} = \underset{\sim}{X}\,\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$$

- Least Squares Estimates: $\hat{\underset{\sim}{\beta}} = (\hat{\beta}_0, \hat{\beta}_1)'$ chosen to minimize SSE
  - $\underset{\sim}{Y} = \underset{\sim}{X}\,\underset{\sim}{\beta} + \underset{\sim}{\varepsilon} \;\rightarrow\; \underset{\sim}{\varepsilon} = \underset{\sim}{Y} - \hat{\underset{\sim}{Y}} \;=\; \underset{\sim}{Y} - \underset{\sim}{X}\,\underset{\sim}{\beta} \;\rightarrow\; \underset{\sim}{\varepsilon}'\underset{\sim}{\varepsilon} = \varepsilon_1^{\,2} + \varepsilon_2^{\,2} + \cdots \varepsilon_n^{\,2} = SSE$
  - $\hat{\underset{\sim}{\beta}} = (\hat{\beta}_0, \hat{\beta}_1)' = \left(\underset{\sim}{X}'\,\underset{\sim}{X}\right)^{-1} \underset{\sim}{X}'\,\underset{\sim}{Y}$
  - Geometry of LSEs:

Let $\underset{\sim}{X}_{3\times 2} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}$ and $\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$