Distributions of functions of RVs

- Suppose X is a RV with known PDF $f(x)$ and CDF $F(x)$
- Let $Y=h(X)$
- Y is a new RV with PDF $g(y)$ and CDF $G(y)$

- The *Distribution Function Technique* is a method for finding $g(y)$ and $G(y)$

  Four Steps:

  1) Start with the definition of the CDF for the *new* RV Y: $G(y) = P(Y \le y)$
  2) Rewrite in terms of the CDF of the *original* RV X: $x = h^{-1}(y)$
  3) Take the derivative to get the PDF of the *new* RV Y: $g(y) = G'(y)$
  4) Transform the Domain of X to get the Domain for Y

Example 1:

$X \sim Uniform(3,7)$

$$f(x) = 1/4 \quad 3 \le x \le 7, z.o.w. \qquad F(x) = \begin{cases} 0 & x < 3 \\ \frac{x-3}{4} & 3 \le x \le 7 \\ 1 & x > 7 \end{cases}$$

$Y = X^3$

Theorem 5.1-1 Let Y have a distribution that is $U(0,1)$. Let $F(x)$ have the properties of a cdf of the continuous type with $F(a) = 0$, $F(b) = 1$, and suppose that $F(x)$ is strictly increasing on the support $a < x < b$, where a and b could be $-\infty$ and ∞, respectively. Then the random variable X defined by $X = F^{-1}(Y)$ is a continuous-type random variable with cdf $F(x)$.

Proof The cdf of X is

$$P(X \le x) = P[F^{-1}(Y) \le x], \qquad a < x < b.$$

Since $F(x)$ is strictly increasing, $\{F^{-1}(Y) \le x\}$ is equivalent to $\{Y \le F(x)\}$. It follows that

$$P(X \le x) = P[Y \le F(x)], \qquad a < x < b.$$

But Y is $U(0,1)$; so $P(Y \le y) = y$ for $0 < y < 1$, and accordingly,

$$P(X \le x) = P[Y \le F(x)] = F(x), \qquad 0 < F(x) < 1.$$

That is, the cdf of X is $F(x)$. □

- Theorem 5.1-1 shows how a sample of observations from a $U(0,1)$ RV can be used to simulate observations from any RV, X, with CDF $F(x)$
  - Start with a $U(0,1)$ value y_i
  - Find the inverse function of the CDF that you want to simulate from $F^{-1}(y)$
  - Transform the y_i to $x_i = F^{-1}(y_i)$ → the x_i are simulated values from a RV with CDF $F(x)$
- Theorem 5.1-1 shows how the expected Quantiles are generated in a Q-Q plot
  - the observed proportions [$p=(i/(n+1)$] play the role of the $Y \sim U(0,1)$ RV
  - $\underline{Q\_expected}= F^{-1}(p_i)$ (e.g., an ***Exponential(3)*** RV below)