1. When was the first recorded “die-shaped” object used?
(a) 3500 BC (b) 1600 BC (c) 2 BC (d) 1492 AD (e) 1904 AD

2. What was it made of?
(a) stone (b) clay (c) kryptonite (d) bone (e) lutefisk

3. How many sides did it have?
(a) 1 (b) 4 (c) 6 (d) 8 (e) 12

4. Where was it discovered?
(a) England (b) China (c) Foxwoods Casino (d) Egypt (e) Babylon

5. Where was the ordinary deck of cards supposed to have been developed?
(a) England (b) China (c) U.S.A. (d) Egypt (e) Babylon

6. When did the study of probability formally begin?
(a) 2nd C. BC (b) 2nd C. AD (c) 8th C. AD (d) 15th C. AD (e) 18th C. AD

<u>Sample Space</u> (***S***): the set of all possible outcomes

<u>Event</u> (E): a collection of outcomes

<u>Ex</u>: Roll a die → ***S*** = {1, 2, 3, 4, 5, 6}

A: # of pips is odd A = {1, 3, 5}
B: # of pips ≥ 3 B = {3, 4, 5, 6}
C: # of pips is even C = {2, 4, 6}

<u>Intersection</u> $(E \cap F)$: The set of elements that belong to both E and F

<u>Union</u> $(E \cup F)$: The set of elements that belong to at least one of E or F

<u>Complement</u> $(\bar{E} \text{ or } E^c)$ All elements of ***S*** that are not in the set E

<u>Algebra of Sets</u>

- Commutative Law $E \cup F = F \cup E$
- Associative Law $(E \cup F) \cup G = E \cup (F \cup G)$
- Distributive Law $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$
- De Morgan’s Laws $(E \cup F)^c =$
  $(E \cap F)^c =$

“Classical”* vs. Relative Frequency Interpretations of Probability

“Classical” Definition of Probability

- Based on equally likely outcomes

- $P(A) = \dfrac{\text{\# ways A can occur}}{\text{\# points in Sample Space}} = \dfrac{n(A)}{n(S)}$

Ex: Roll a die → $\boldsymbol{S}$ = {1, 2, 3, 4, 5, 6}

A: odd # of pips   A = {1, 3, 5}
B: # of pips ≥ 3   B = {3, 4, 5, 6}
C: even # of pips   C = {2, 4, 6}

P(A) =
P(B) =
P(A∩B) =
P(A∩C) =

Relative Frequency Definition of Probability

- If $n(E)$= # of times E occurs in n repetitions of the experiment
  Then, $P(E) = \lim_{n\to\infty} \dfrac{n(E)}{n}$

Axioms of Probability

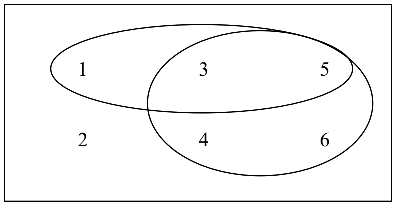
1. $0 \le P(E) \le 1$
2. $P(S) = 1$
3. If $E_1, E_2, and\ E_3$ are all mutually exclusive
   Then, $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$

- Working with the Axioms

$$S = E \cup E^c$$
$$P(S) = P(E \cup E^c)$$
$$1 = P(E) + P(E^c)$$

Ex: Roll a die → $S = \{1, 2, 3, 4, 5, 6\}$

A: odd # of pips   A = {1, 3, 5}
B: # of pips ≥ 3   B = {3, 4, 5, 6}



$$A \bigcup B = \{1,3,4,5,6\} \to P(A \bigcup B) = \frac{5}{6}$$

$$P(A \bigcup B) = P(A) + P(B) = \frac{3}{6} + \frac{4}{6} = \frac{7}{6} \text{ ??}$$

General Addition Rule for Probability

$$P(A \bigcup B) =$$

Multiplication Principle:

- If E_1 has n_1 outcomes and E_2 has n_2 outcomes,
  there are $n_1\, n_2$ possible outcomes for the composite experiment (E_1 and E_2)

  → There are ***n!*** permutations of ***n*** objects

Permutations of ***n*** objects taken ***r*** at a time: ${}_n\mathbf{P}_r$ = n (n-1) (n-2) … (n-r+1)

- The # of ways to select ***r*** items from ***n*** items where order matters

Combinations of ***n*** objects taken ***r*** at a time: ${}_n\mathbf{C}_r = \dfrac{n(n-1)\cdots(n-r+1)}{r!} = \dfrac{n!}{(n-r)!r!} = \dbinom{n}{r}$

- The # of ways to select ***r*** items from ***n*** items where order DOES NOT matter
- Distinguishable permutations of ***n*** objects with ***r*** of one type and ***(n-r)*** of another

Binomial Coefficients:

$$(x+y)^0 = 1$$
$$(x+y)^1 = 1x + 1y$$
$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$
$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

Pascal's Triangle:

```
row 0                     1
row 1                   1   1
row 2                 1   2   1
row 3               1   3   3   1
row 4             1   4   6   4   1
row 5           1   5   10  10   5   1
```

Suppose the pound has 8 Golden Retriever puppies and 6 Shepherd puppies

(a) In how many way can you select 2 Goldens and 3 Shepherds?

(b) Answer (a) if there are 2 female Shepherds and you can't choose both females.

Distinguishable Permutations (continued)

- Suppose there are $\boldsymbol{n}$ items with $\boldsymbol{n_1}$ alike, $\boldsymbol{n_2}$ alike, … , and $\boldsymbol{n_s}$ alike.
  There are $\dfrac{n!}{n_1!n_2!\cdots n_s!}$ distinguishable permutations of the $\boldsymbol{n}$ items.

- When $n_1 + n_2 + \cdots + n_s = n$, the terms $\dfrac{n!}{n_1!n_2!\cdots n_s!} = \dbinom{n}{n_1, n_1, \cdots n_s}$ are called Multinomial Coefficients since they are the coefficients in the expansion of the multinomial, $(x_1 + x_2 + \cdots + x_s)^n$

"Non-negative Ball and Urn Model"

- Set up: $\boldsymbol{r}$ balls (represented by $\boldsymbol{r}$ 0's)
  $\boldsymbol{n}$ urns (represented by $\boldsymbol{n-1}$ |'s)

  sampling the $\boldsymbol{n}$ urns a total of $\boldsymbol{r}$ times *with replacement* to put a ball in each time

- The # of *unordered* samples of size $\boldsymbol{r}$ that can be selected out of $\boldsymbol{n}$ objects *with replacement* is given by
  $\dbinom{n-1+r}{r} = \dfrac{(n-1+r)!}{r!(n-1)!}$ (i.e., the # of *distinguishable* permutations of 0's & |'s)

- The formal statement:
  Let $x_1, x_2, \cdots, x_n$ be non-negative integers with $x_1 + x_2 + \cdots + x_n = r$ (representing the # of balls in each urn)

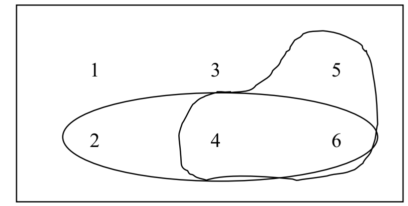
  There are $\dbinom{n-1+r}{r} = \dfrac{(n-1+r)!}{r!(n-1)!}$ distinct sets of non-negative integers $(x_1, x_2, \cdots, x_n)$ satisfying $x_1 + x_2 + \cdots + x_n = r$.

Ex: (non-negative ball & urn model)

In how many ways can 6 identical items be distributed among 3 different stores?

Conditional Probability

Ex: Roll a die → $\boldsymbol{S} = \{1, 2, 3, 4, 5, 6\}$



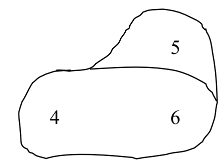
C: even # of pips $C = \{2, 4, 6\}$
D: # of pips ≥ 4 $D = \{4, 5, 6\}$

$P(C) = \frac{3}{6}, \; P(D) = \frac{3}{6}$

What if we knew that event $\boldsymbol{D}$ occurs?

What is the probability of $\boldsymbol{C}$ if $\boldsymbol{D}$ occurs?

- If event $\boldsymbol{D}$ occurs, then it becomes our *reduced sample space*:



$$P(C \mid D) = \frac{\text{\# ways } C \cap D \text{ can occur}}{\text{\# points in } \textit{Reduced Sample Space}}$$
$$= \frac{n(C \cap D)}{n(D)}$$
$$= \frac{P(C \cap D)}{P(D)}$$

Definition of Conditional Probability

If $P(B) > 0$ then $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

<u>Sample Space (*S*) for rolling 2 die</u>: ***n(S) = 36***

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

What is the probability that the sum on the two die is 6 if they land on different numbers?

Event A: the sum is 6
Event B: the die land on different numbers → P(A | B)

Events A and B are *<u>independent</u>* if $P(A \cap B) = P(A) \bullet P(B)$

Independence for A & B → conditional probabilities are the same as unconditional probabilities

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \bullet P(B)}{P(B)} = P(A)$$

<u>Example</u>: A card is drawn at random from an ordinary deck of cards

A: event that an ***Ace*** card is drawn
C: event that a ***Club*** card is drawn
B: event that a ***Black*** card is drawn

Are events ***A*** & ***C*** independent?

Are events ***B*** & ***C*** independent?