Technical Appendix

Appendix C

Parameters of the labor income function:

The parameters of the income functions satisfy the following conditions. Recall that $z-s$ represents age. The labor income of the poor is lower than the labor income of the rich:

$$a_P e^{-\alpha_P(z-s)} < a_R e^{-\alpha_R(z-s)}$$

Aggregate effective labor supply:

Aggregate effective labor supply for an individual of type i is described by:

$$\begin{aligned}\tilde{N}^i(t) &= \int_{-\infty}^{t} \frac{1}{\lambda_P+\lambda_R} e^{-\frac{1}{\lambda_i}(t-s)} \tilde{N}^i(s,t) \\ &= \int_{-\infty}^{t} \frac{1}{\lambda_P+\lambda_R} e^{-\frac{1}{\lambda_i}(t-s)} [a_i e^{-\alpha_i(t-s)}] \tilde{N}^i(t) ds\end{aligned}$$

Therefore, implicitly, a_i, α_i satisfy:

$$\int_{-\infty}^{t} \frac{1}{\lambda_P+\lambda_R} e^{-\frac{1}{\lambda_i}(t-s)} [a_i \exp^{-\alpha_i(t-s)}] ds = \frac{\lambda_i}{\lambda_P+\lambda_R}$$

$$\frac{1}{\lambda_P+\lambda_R}\left(\frac{a_i}{\frac{1}{\lambda_i}+\alpha_i}\right) = \frac{\lambda_i}{\lambda_P+\lambda_R}$$

Appendix D

Aggregate human wealth

For a rich individual, it is given by:

$$H^R(t) = \int_{-\infty}^{t} \frac{1}{\lambda_P+\lambda_R} e^{-\frac{1}{\lambda_R}(t-s)} H^R(s,t) ds$$

where $H^R(s,t)$ denotes individual human wealth and equals:

$$H^R(s,t) = \int_{t}^{+\infty} e^{-\int_t^z \left[\hat{r}(v)+\frac{1}{\lambda_R}\right]dv} \hat{\omega}^R(z) \tilde{N}^R(s,z) dz$$

Using the age distribution of labor income, individual human wealth is rewritten:

$$H^R(s,t) = \int_{t}^{+\infty} \frac{1}{\lambda_P+\lambda_R} e^{-\int_t^z \left[\hat{r}(v)+\frac{1}{\lambda_R}\right]dv} \hat{\omega}^R(z) [a_R e^{-\alpha_R(z-s)}] E^R(z) N^R(z) dz$$

After rearranging the integrals, aggregate human wealth is given by:

$$H^R(t)=\int_{-\infty}^{t}\frac{1}{\lambda_P+\lambda_R}e^{-\frac{1}{\lambda_R}(t-s)}\left[a_R e^{-\alpha_R(t-s)}\right]$$
$$\left\{\int_{t}^{+\infty}e^{-\int_t^z\left[\hat{r}(v)+\frac{1}{\lambda_R}\right]dv}\hat{\omega}^R(z)\left[\exp^{-\alpha_R(z-t)}\right]E^R(z)N^R(z)dz\right\}ds$$

I reorganize the integrals as:

$$H^R(t)=\int_{-\infty}^{t}\frac{a_R}{\lambda_P+\lambda_R}e^{-(\frac{1}{\lambda_R}+\alpha_R)(t-s)}\left[\int_{t}^{+\infty}e^{-\int_t^z\left[\hat{r}(v)+\frac{1}{\lambda_R}+\alpha_R\right]dv}\hat{\omega}^R(z)E^R(z)N^R(z)dz\right]ds$$

with:

$$\lim_{z->+\infty} e^{-\int_t^z\left[\hat{r}(v)+\frac{1}{\lambda_R}+\alpha_R\right]dv}\hat{\omega}^R(z)E^R(z)N^R(z)=0$$

and:

$$\int_{-\infty}^{t}\frac{a_R}{\lambda_P+\lambda_R}e^{-(\frac{1}{\lambda_R}+\alpha_R)(t-s)}=1$$

$$\frac{a_R}{\lambda_P+\lambda_R}\frac{1}{\frac{1}{\lambda_R}+\alpha_R}=1$$

The term in braces is independent of s. Differentiating with respect to time yields:

$$\dot{H}^R(t)=\left(\hat{r}(t)+\frac{1}{\lambda_R}+\alpha_R\right)H^R(t)-\hat{\omega}^R(t)E^R(t)N^R(t)$$

Appendix E

In the steady state,
When $\dot{C}^R(t)=0$,

$$(\hat{r}+\alpha_R-\delta)C^R=\left(\delta+\frac{1}{\lambda_R}\right)\left(\frac{1}{\lambda_R}+\alpha_R\right)W^R$$

As a result:

$$\delta-\alpha_R<\hat{r}$$

The concavity of F implies that:

$$F(K)>rK$$

which gives:

$$C^R + C^P + rT_k K > rK$$

or equivalently:

$$\left(\delta + \frac{1}{\lambda_R}\right)(K^R + H^R) > \hat{r}K - C^P$$

$$\left(\delta + \frac{1}{\lambda_R} - \hat{r}\right)K^R + \left(\delta + \frac{1}{\lambda_R}\right)H^R + (C^P - \hat{r}K^P) > 0 \tag{22}$$

For this inequality to hold, two sufficient conditions are:

$$\hat{r} \leqslant \delta + \frac{1}{\lambda_R} \tag{23}$$

and:

$$C^P - \hat{r}K^P \geqslant 0 \tag{24}$$

Since: $\frac{1}{\lambda_R} \leqslant \frac{1}{\lambda_P}$, we also get:

$$\hat{r} \leqslant \delta + \frac{1}{\lambda_P} \tag{25}$$

If the concavity condition (23) is satisfied, then (25) is satisfied.

Furthermore, I show that if the concavity condition (23) is satisfied, (24) is always satisfied in the scenarios that we consider. If the poor do not annuitize, with $\hat{r} \leqslant \delta + \frac{1}{\lambda_P}$, then the poor desire to consume more than their income and do not save. In the presence of no-borrowing constraints the no-borrowing constraint binds and the poor consume their entire disposable income. As a result, $K^P = 0$. Therefore, $C^P > 0$. The two sufficient conditions for the concavity of F are satisfied. If the poor annuitize, aggregate consumption by the poor is written in similar way to aggregate consumption by the rich. This implies that (24) is satisfied and inequality (22) is rewritten as:

$$\left(\delta + \frac{1}{\lambda_R} - \hat{r}\right)K^R + \left(\delta + \frac{1}{\lambda_R}\right)H^R + \left(\delta + \frac{1}{\lambda_P} - \hat{r}\right)K^P + \left(\delta + \frac{1}{\lambda_P}\right)H^P > 0$$

The only sufficient condition for the concavity of F is therefore condition (23).

Appendix F

Conditions for the strict concavity of V

Recall that V is described as follows:

$$V(C(z), 1 - N(z)) \;=\; \max_{\{C(z-n,z),N(z-n,z)\}_{n=0}^{+\infty}} \int_0^{+\infty} \begin{bmatrix} e^{-\left(\frac{1}{\lambda_P}\right)n} U(C^P(z-n,z), 1 - N^P(z-n,z)) \\ + e^{-\frac{1}{\lambda_R}n} \upsilon(C^R(z-n,z), 1 - N^R(z-n,z)) \end{bmatrix} e^{(\rho-\delta)n} dn$$

Typically, the value function inherits concavity from the utility function. U is assumed to be increasing in consumption and leisure, strictly concave and two times continuously differentiable. Therefore:

$$\frac{\partial^2 U}{\partial C^P(z)} < 0$$

$$\frac{\partial^2 U}{\partial L^P(z)} < 0$$

I now need to determine the conditions for the strict concavity of υ.

Using:

$$\upsilon_{C^R(z-n,z)} = \left(1 + \chi(z-n) + \chi(z-n)\xi^{C^R(z-n,z)}\right) U_{C^R(z-n,z)}$$

With:

$$\xi^{C^R(z-n,z)} = \frac{U_{C^R(z-n,z)C^R(z-n,z)} C^R(z-n,z) + U_{L^R(z-n,z)C^R(z-n,z)} L^R(z-n,z)}{U_{C^R(z-n,z)}}$$

and:

$$\upsilon_{L^R(z-n,z)} = \left(1 + \chi(z-n) + \chi(z-n)\xi^{L^R(z-n,z)}\right) U_{L^R(z-n,z)}$$

with

$$\xi^{L^R(z-n,z)} = \frac{U_{C^R(z-n,z)L^R(z-n,z)} C^R(z-n,z) + U_{L^R(z-n,z)L^R(z-n,z)} L^R(z-n,z)}{U_{L^R(z-n,z)}}$$

I obtain:

$$\frac{\partial^2 \upsilon}{\partial C^R(z)} = \frac{\partial \upsilon_{C^R(z-n,z)}}{\partial C^R(z)}$$

$$= \chi(z-n)\frac{\partial \xi^{C^R(z-n,z)}}{\partial C^R(z)} U_{C^R(z-n,z)} + \left(1 + \chi(z-n) + \chi(z-n)\xi^{C^R(z-n,z)}\right)\frac{\partial U_{C^R(z-n,z)}}{\partial C^R(z)}$$

$$= \chi(z-n)\frac{\partial \xi^{C^R(z-n,z)}}{\partial C^R(z)} U_{C^R(z-n,z)} + \left(1 + \chi(z-n) + \chi(z-n)\xi^{C^R(z-n,z)}\right)\frac{\partial^2 U(\bullet)}{\partial C^R(z)}$$

Therefore,

$$\frac{\partial^2 \upsilon}{\partial C^R(n)} < 0 \text{ if } \chi(z-n)\frac{\partial \xi^{C^R(z-n,z)}}{\partial C^R(z)} U_{C^R(z-n,z)} + \left(1 + \chi(z-n) + \chi(z-n)\xi^{C^R(z-n,z)}\right)\frac{\partial^2 U(\bullet)}{\partial C^R(z)} < 0$$

In a similar way,

$$\frac{\partial^2 \upsilon}{\partial L^R(n)} < 0 \text{ if } \left\{\chi(z-n)\frac{\partial \xi^{L^R(z-n,z)}}{\partial L^R(z)} U_{L^R(z-n,z)} + \left(1 + \chi(z-n)\chi(z-n)\xi^{L^R(z-n,z)}\right)\frac{\partial^2 U(\bullet)}{\partial L^R(z)}\right\} < 0$$

If those two last conditions are satisfied, υ is concave.