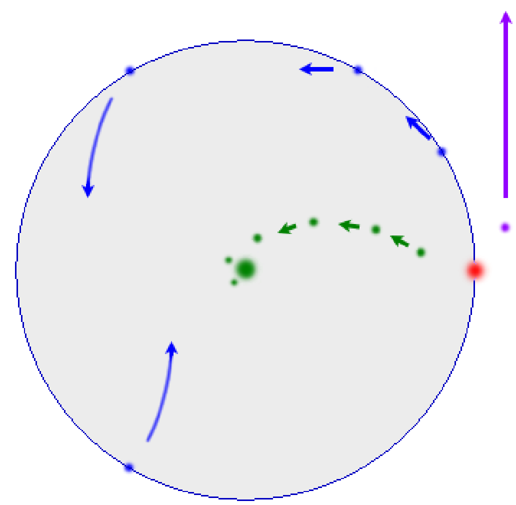
EZ Julia Sets

by Michael Sargent - March 2016

Most of the material in this article, except for the Herman rings, has been adapted from Chapters 13 and 14 of *Chaos and Fractals - New Frontiers of Science* by Heinz-Otto Peitgen, Hartmut Jűrgens and Dietmar Saupe (Springer-Verlag 1992).

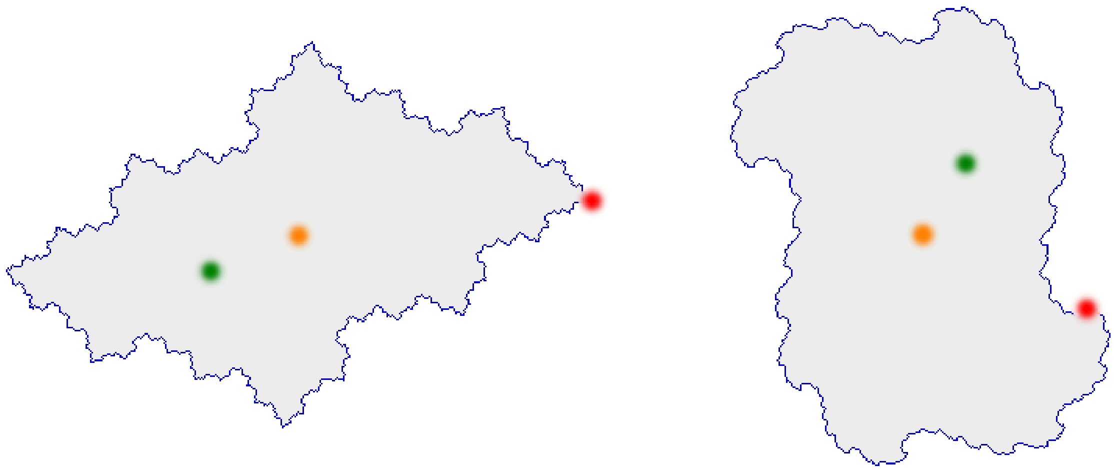
All of the images were rendered by custom software programmed by the author.

The Julia formula $z_{n+1} = z_n^2 + c$ simplifies to $z_{n+1} = z_n^2$ when c = 0. Therefore, for any initial value z_0 whose absolute value is 1, each subsequent iteration also yields a point whose absolute value is 1. These points are located on the *unit circle* which constitutes the Julia set of c = 0. Eventually the iterated points reach either a repeating periodic cycle or the *fixed point* 1 + 0i. (Iteration of a fixed point yields that same fixed point indefinitely.) This fixed point is called *repelling* because iteration of any point z_0 which is close to it yields points which are further away from it. Since these close points can go in any direction, the dynamics around the J-set are *chaotic*. Because iteration of any point of the J-set itself yields only other points of the set, J-sets are called *invariant*.

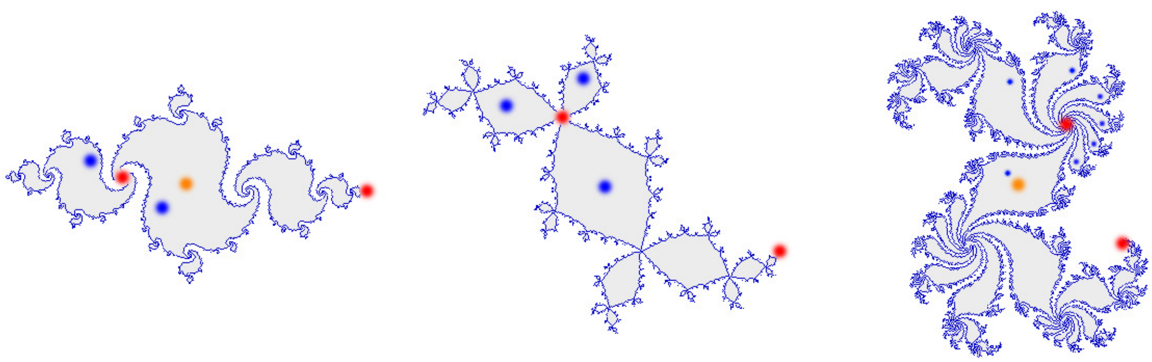


If $|z_0| < 1$, the iterated points eventually reach a different fixed point, 0 + 0i. All those points are obviously inside the unit circle, and are not part of the J-set; they constitute the interior, or *prisoner*, set. This fixed point is called *attracting* because iteration of any point z_0 which is close to it yields points which are closer to it. When c = 0, the attracting fixed point is also the *critical point*, whose derivative (2z) is 0. When this is true, the fixed point is called *superattractive*. The critical point is the only point for which $z^2 + c = c$. For more complicated formulae, the critical point is not always 0.

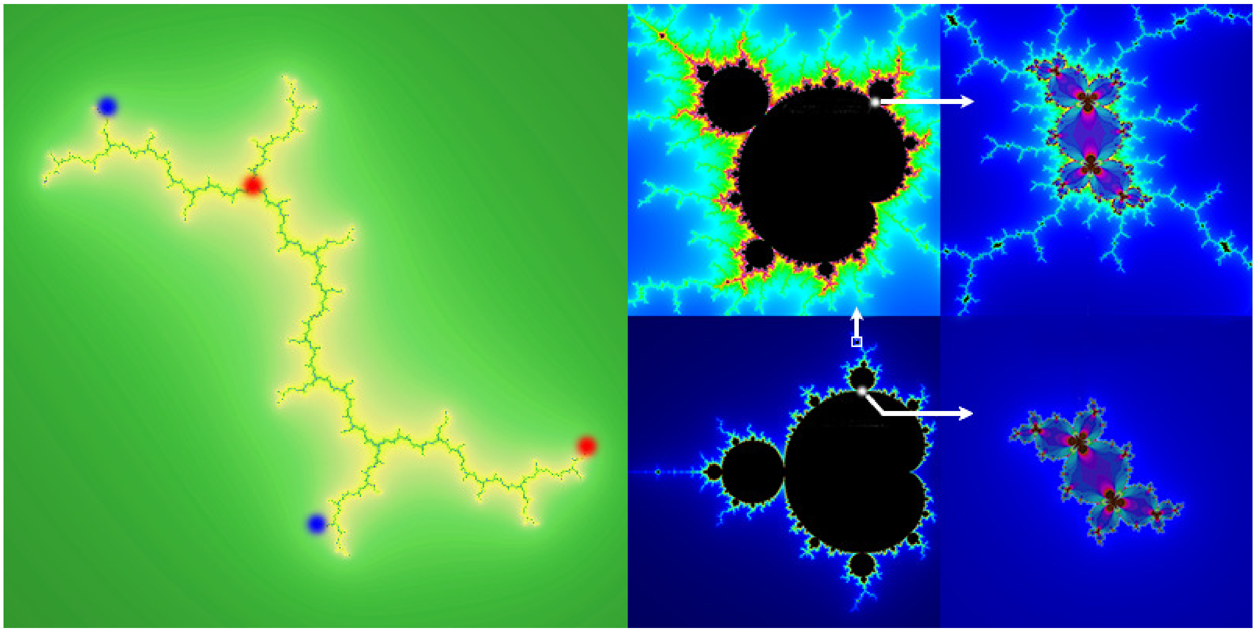
If $|z_0| > 1$, the absolute values of the iterated points become larger than max(|c|, 2), at which point they are recognized as escaping to another attractor: infinity. These points constitute the *escape* set. Thus, the J-set is the simultaneous boundary of the prisoner and escape sets.



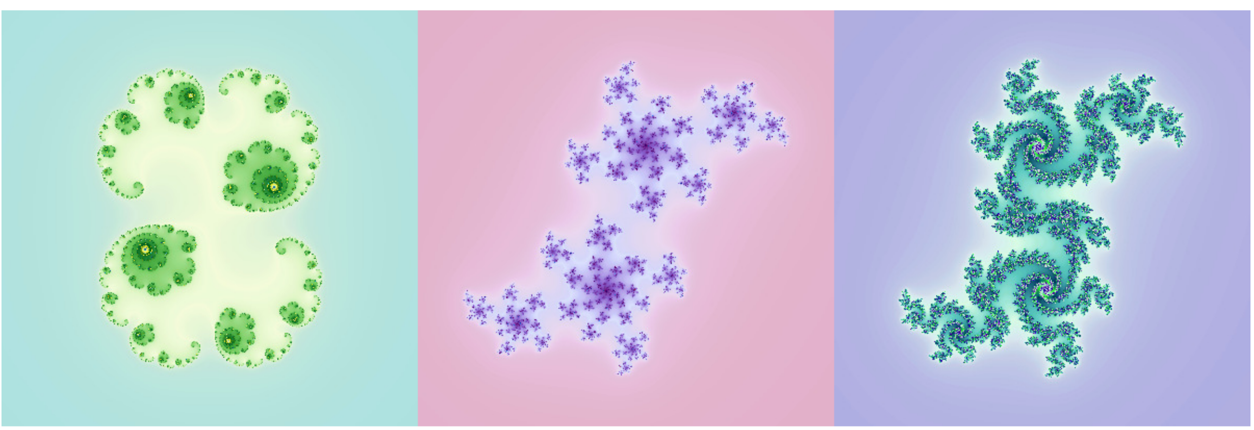
All points c which are associated with attracting fixed points come from within the boundary of the main cardioid of the Mandelbrot set. These fixed points are no longer the critical points. We continue to find second, repelling fixed points in the J-sets, which are now symmetrical and self-similar deformations of the unit circle.



Points from the buds attached to the cardioid of the M-set, or subsidiary buds, are associated with attracting cycles of points with the same periods as those of the buds, rather than single attracting fixed points. Instead, these points surround one of the fixed points. This fixed point is located at a junction of a group of self-similar curves, equal in number to the period. The J-sets are composed of infinite numbers of such groups of self-similar curves. Both fixed points are now repelling, since points close to them iterate progressively closer either to one of the periodic attracting points or infinity, and away from the fixed points. The central image demonstrates a superattractive case, where one of the attractive periodic points also is the critical point 0 + 0i. These occur when c is in the center of the M-set bud. When a parameter c of the M-set is located exactly at the *attachment* of a bud to a larger bud, it has a different type of fixed point, neither attracting nor repelling, which is called *indifferent*, and which will be described further below.



A different type of J-set is associated with points c from the filaments at the outermost edges of the buds of the M-set. These sets are known as *dendrites* and have no interior prisoner sets. Therefore they are essentially invisible on a computer screen and must be displayed indirectly, with level set, inverse iteration or distance estimation methods that outline the J-sets with variously-colored escape points near the boundaries of the J-sets themselves. Embedded within M-set filaments are infinite numbers of regions which resemble somewhat distorted miniature M-sets. A point c from one of these miniatures yields a dendritic J-set with an infinite number of embedded miniature J-sets which closely resemble the J-set which would be obtained from a corresponding point of the main M-set.



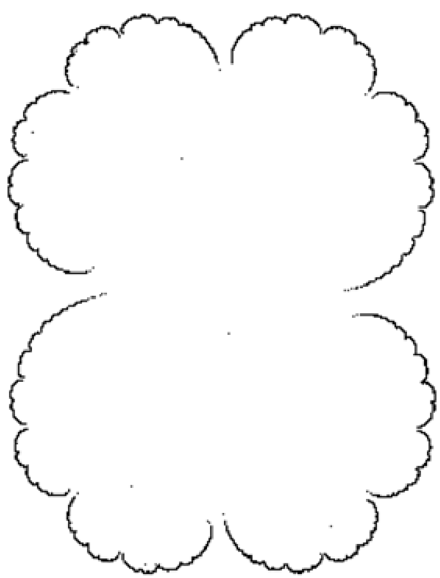
The J-sets described so far are *connected* within a single curve. While the M-set is typically defined as the set of all points c in the complex parameter plane which do not escape to infinity when iterated, it also can be defined as the set of all points whose J-sets are connected. The J-sets of points in the *complement* of the M-set are *disconnected*, composed of infinite numbers of separate self-similar curves, referred to as *Cantor sets* or *Fatou dusts*. As with dendrites, graphic images of these sets are best displayed indirectly.

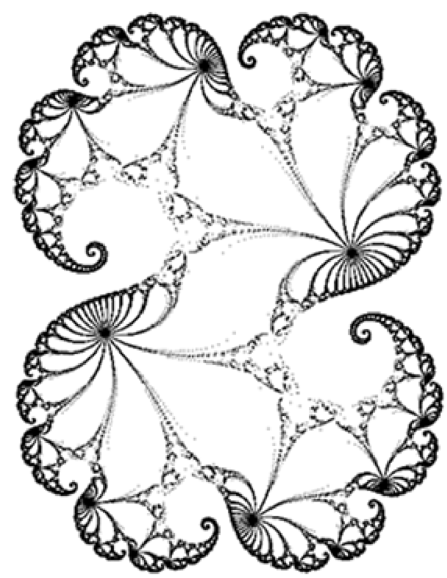
Since fixed points by definition yield themselves when iterated, this means that if z_n is a fixed point, $z_n^2 + c = z_n$. This can be expressed as $z_n^2 - z_n + c = 0$, which is in a format that can be solved using the standard quadratic formula, yielding the following calculation for the fixed points of any J-set:

$$z = ½ \pm \sqrt{1/4 - c}, \quad \text{or}$$
$$z = ½ \pm \sqrt{1 - 4c}\ / 2$$

The derivative ($2z$) is therefore $1 \pm \sqrt{1 - 4c}$. If the absolute value of the derivative is greater than 1, the fixed point is repelling and is a member of the J-set. If it is less than 1, the fixed point is attracting and is a member of the prisoner set. If it equals 1, the fixed point is indifferent. This provides a useful method for drawing a connected Julia set: the inverse iteration method.

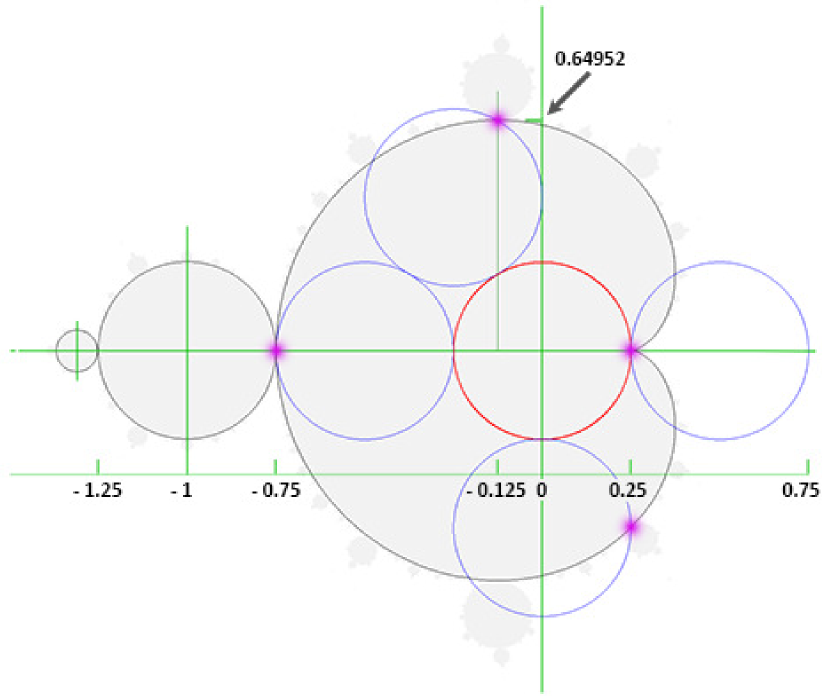
If a fixed point is repelling for standard forward iteration, then it should be attracting for inverse iteration. Thus, any random initial point z_0 will be attracted to the Julia set when $z_{n+1} = \pm \sqrt{z_n - c}$, and further iterations will land on other points of the J-set. To avoid having to discard a number of initial iterations, we can start at a repelling fixed point, obtained with the above calculations. If c is in the main cardioid, one point will be repelling; in the other buds, both will be.





Since there are 2 solutions to $\sqrt{z_n - c}$, at each iteration one of the solutions is chosen at random for the next iteration. The left image shows that this chaos game approach often leads to incomplete images, because iterated preimages along the resulting branching tree are not distributed uniformly. Some points are "hit" frequently and others are hit only rarely. Raster graphics require rounding when points are transformed to pixel values, and further floating point rounding by the computer processor also contributes to this problem. Peitgen and others have devised various modified inverse iteration methods (MIIM), which improve the quality of the images, as shown on the right.

Cardioids are a sub-set of cycloids, which are curves drawn by a point of a circle rolling around the circumference of another fixed circle. If the drawing point is on the circumference of the moving circle, the curve is a cycloid; if it is inside the moving circle, the curve is a trochoid. If the moving circle is outside of the fixed circle, the prefix epi- is applied; if it is inside the fixed circle, the prefix hypo- is applied. Cardioids are epicycloids in which the radii a and b of both circles are equal. If that radius is 0.25, we have the M-set cardioid.



The epicycloid formula is:

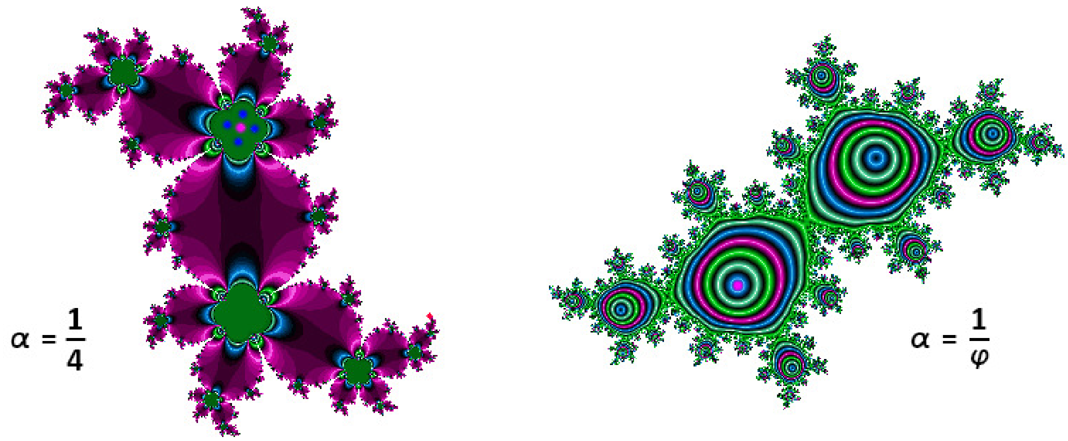
$$x = (a + b) * \cos (\Phi) - b * \cos ((a / b) + 1) * \Phi)$$
$$y = (a + b) * \sin (\Phi) - b * \sin ((a / b) + 1) * \Phi)$$

for $0 \leq \Phi < 2\pi$

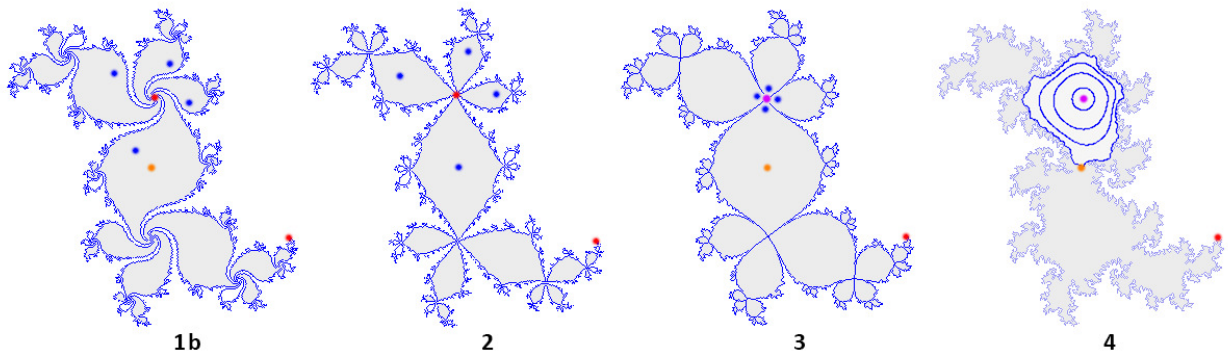
With the radii a and b = 0.25, the M-set cardioid formula reduces to:

$$x = ((\cos (\Phi) / 2) - (\cos (2.0 * \Phi) / 4))$$
$$y = ((\sin (\Phi) / 2) - (\sin (2.0 * \Phi) / 4))$$

which yields a point c = x + y i on the cardioid boundary for any angle Φ . The fixed points for the J-set of c can be calculated as described above. Taking their derivatives reveals that one of them is repelling, and the other one, whose derivative is 1, is indifferent. Two types of J-sets result, depending on the nature of c.



Φ can be expressed as $2 \pi \alpha$. If ($\alpha = j / k$) is rational, $c = x[\Phi] + y[\Phi]\, i$ is the attachment point of a bud of period k. The fixed point of the J-set of c with a derivative of 1 is called *rationally indifferent* or *parabolic*, and it is part of the J-set. For other values of c, in which α is irrational, the fixed point with a derivative of 1 is called *irrationally indifferent* and it is part of the prisoner set. It is the center of a *Siegel disk,* which is formed from a set of attracting invariant curves. The most well-known case of a Siegel disk occurs when $\alpha = 1 / \varphi$, where φ is the golden mean = $(\sqrt{5} + 1) / 2$. This is the irrational number to which sequences of rational numbers converge most slowly.



We can summarize by describing the Sullivan classification:

1) Attractive cases: a) The interior of the cardioid is the set of all c for which one of the fixed points is attractive. b) The buds comprise the set of all c for which both fixed points are repelling, and the attractor is a periodic cycle surrounding one of the (junctional) fixed points.

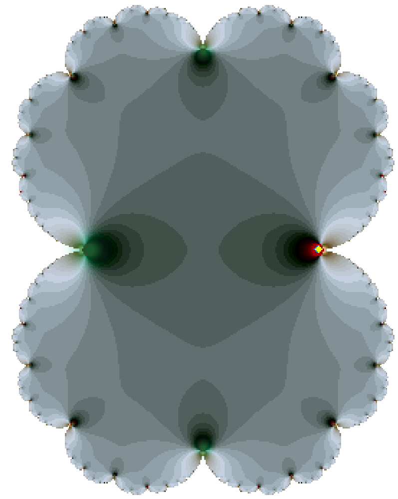
2) Superattractive cases: Both regions from #1 can include cases for which (one of) the (periodic) attractive point(s) is the critical point.

The boundary of the cardioid includes those c for which one of the fixed points is indifferent.

3) Parabolic cases: If the point is rationally indifferent, the point is part of the J-set, surrounded by a periodic attracting cycle.

4) Siegel disks: If the point is irrationally indifferent, the point is part of the prisoner set, which contains a Siegel disk of attracting invariant curves surrounding the point.

If one fixed point is attractive, the other must be repelling. If one fixed point is indifferent, the other must be repelling, unless the fixed points are identical ($c = 0.25 + 0i$, fixed points = 0.5, derivatives = 1, attractors = 0.49967).



There is a fifth category in the Sullivan classification: Herman rings. They are similar to Siegel disks, but are found in rational functions, and do not occur with polynomial functions. Buried in Michael Robert Herman's 1979 treatise, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations", Publications Mathématiques de l'IHÉS (49)* - 229 pages in French - is this formula of a rational function, which yields Herman rings with the "proper" parameters:

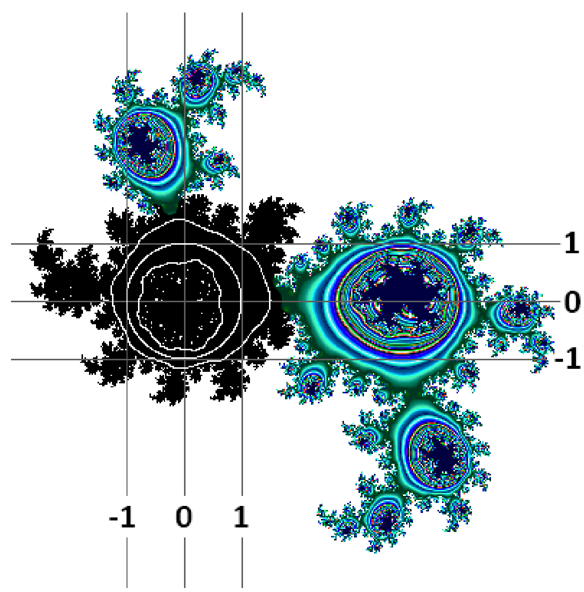
$$z = e^{2\pi\varphi i} * z * \frac{(z - a) * (1 - z * \mathrm{conj}(b))}{(z - b) * (1 - z * \mathrm{conj}(a))}$$

with φ irrational, $|a| < 1$ and $|b| < 1$, *e.g.* $\sqrt{5}$, $(0 - 0.1i)$, $(0.28 - 0.1i)$

In this example, φ is the golden mean:

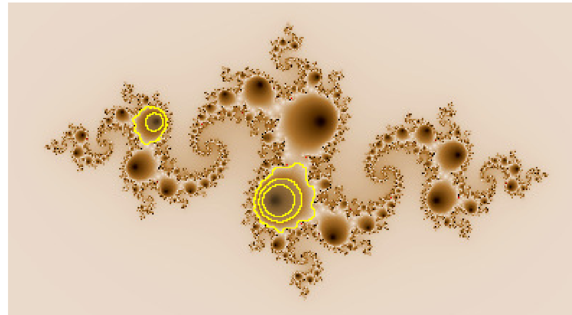
$$z = e^{\pi\varphi i} * z^2 * \frac{(z - 4)}{(1 - 4z)}$$

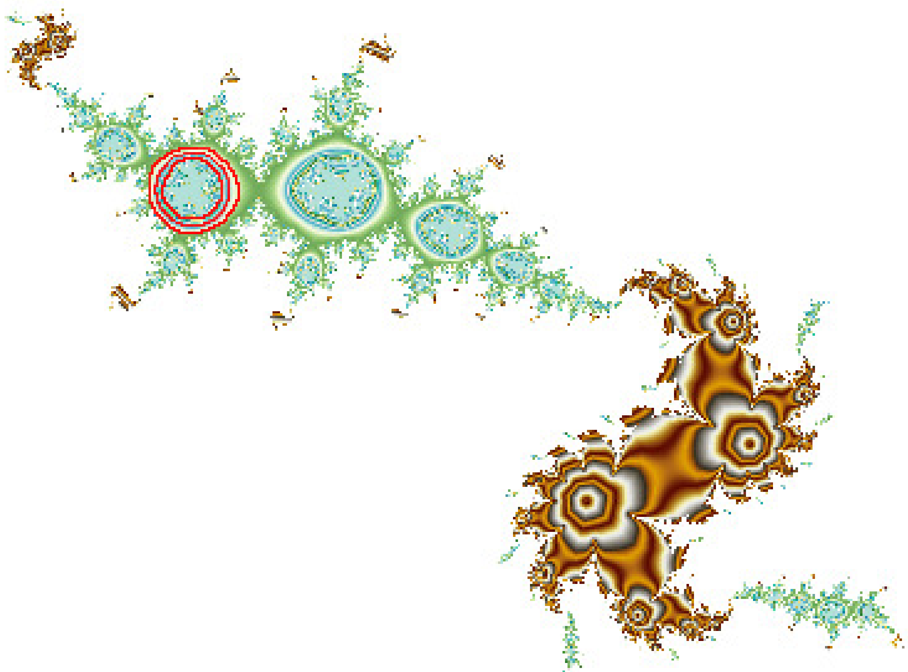
The white circle within the Herman ring is the unit circle.



Here is an example of Herman rings with a period of 2. The formula is from Mitsuhiro Shishikura:

$$z = z^2 - 1 - e^{2 \pi \varphi i} * / 4$$





Finally, this strange image consists of alternating sets of Herman rings and parabolic cases. The formula is from Wikipedia contributor "Wangfei math". 0.618... = (sqrt(5) - 1) / 2.

$$a = 0.25 + 0i \qquad b = 0.0405353 - 0.0255082i$$

$$z = e^{2 \pi i * 0.61803399} * z^3 * \frac{(1 - \mathrm{conj}(a) * z) * (1 - \mathrm{conj}(b) * z)}{(z - a) * (z - b)}$$

Herman's treatise: http://archive.numdam.org/article/PMIHES_1979__49__5_0.pdf

Mitsuhiro Shishikura, *On the quasiconformal surgery of rational functions*. Ann. Sci. Ecole Norm. Sup. (4) **20** (1987), no. 1, 1–29 (http://archive.numdam.org/article/ASENS_1987_4_20_1_1_0.pdf)
Mitsuhiro Shishikura, *Surgery of complex analytic dynamical systems*, in "Dynamical Systems and Nonlinear Oscillations", World Scientic Advanced Series in Dynamical Systems, 1, World Scientic, 1986, 93–105 (http://repository.kulib.kyoto-u.ac.jp/dspace/bitstream/2433/99208/1/0574-13.pdf)