Example 4

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a ring A, $X \neq \emptyset$, and R the ring functions from $X \to A$.

$f \in R$, $c \in X$ $\quad E_c(f) = f(c)$

$E_c(f+g) = (f+g)(c) = f(c) + g(c) = E_c(f) + E_c(g)$ Same goes for $*$

Find $\ker(E_c)$. $\quad \ker(\varphi) = \varphi^{-1}(0)$

$$\ker(E_c) = \{ f \in R \mid E_c(f) = 0 = f(c) \}$$

Show that E_c is surjective

$f(x) = a$, $a \in A$. Clearly, $f(x) \in R$

By the first homomorphism thrm, we know $R/\ker(E_c) \cong E_c(R)$

$$\Downarrow$$

$$R/\ker(E_c) \cong A$$

Ex: Let $X = [0,1]$ and $A = \mathbb{R}$. So, $R/\ker(E_c) \cong \mathbb{R}$

The $\ker(E_c)$ is the ideal of all cont. func. s.t. $f(c) = 0$ (or $f(c)$ crosses the x-axis)

Example 5

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$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$

Let $\varphi : R[x] \to R$ by $p(x) \to p(0)$

Find $\ker(\varphi) = \{ p(x) \in R[x] \mid p_0 = 0 \}$

Let $\psi : R \to S$. We get a third homomorphism via comp. $\psi(\varphi) : R[x] \to S$

Ex: Consider $\psi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ by $p \to p \pmod 2$

$\varphi : \mathbb{Z}[x] \to \mathbb{Z}$

So, $\psi(\varphi(x)) = p(0) \pmod 2 = p_0 \pmod 2$ $\to$ $\ker(\psi(\varphi)) = \{ p(x) \in \mathbb{Z}[x] \mid p_0 \in \mathbb{Z}/2\mathbb{Z} \}$

Example 6

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M_n is not commutative

Let $n \in \mathbb{Z}$ s.t. $n \geq 2$

show that if J is an ideal of R (ring), then $M_n(J)$ is a two-sided ideal of $M_n(R)$

$a \in M_n(J)$, $b \in M_n(R)$. Is $a \cdot b \in M_n(J)$ and $b \cdot a \in M_n(J)$

$a \cdot b = a_{00} \cdot b_{00} + a_{10} \cdot b_{01} + a_{20} \cdot b_{02} + \ldots\ldots + a_{n0} \cdot b_{0n}$, so $(ab)_{00} \in J$

$a_{r0} \cdot b_{r0} \in J$

Sim. $ba \in M_n(J)$

So, $M_n(J)$ is a two-sided ideal of $M_n(R)$

$R \to R/J$

$q \in J$. $q \to 0$

Let $\varphi : M_n(R) \to M_n(R/J)$. Find $\ker(\varphi) = M_n(J)$

$\underline{a} \in M_n(J)$. $a = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \to a \pmod J = \begin{bmatrix} a_{00} & - \\ - & \cdots \end{bmatrix} \pmod J = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\ker(\varphi) = M_n(J)$

$n = 3$, $R = \mathbb{Z}$. Clearly, $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$, so $M_n(2\mathbb{Z})$ is a two-sided ideal of $M_n(\mathbb{Z})$

$$\begin{bmatrix} 4 & 2 & -6 \\ 0 & 8 & -2 \\ 0 & 0 & 0 \end{bmatrix} \pmod 2 = [\,0\,] \pmod 2$$