Extremal Graph Theory Lecture 1 9/1/21

Connectivity. Graph $G(V, E)$

$V(G)$ is a set of vertices.

$E(G)$ is a set of edges $\subseteq \binom{V}{2}$.

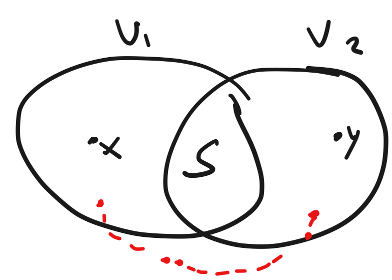
If $V = V_1 \cup V_2$ and $S = V_1 \cap V_2$ then

$|S| = k$

we say S is a k-separator in G if there is no edge from $V_1 \setminus S$ to $V_2 \setminus S$.

If $X \subseteq V_1 \setminus S$ $Y \subseteq V_2 \setminus S$

then S separates X from Y.



Same for vertices x, y.

Then $\kappa(G) = \min\{|G| - 1, k \mid G \text{ has a } k\text{-separator}\}$.

vertex-connectivity. $|G| = |V(G)|$ $\|G\| = |E(G)|$.

$\lambda(G)$ same for removing edges.

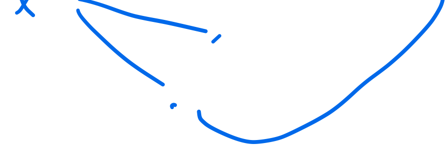
Prop

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

minimum degree of G.

Local vx connectivity:

$k(x,y)$ = min nr of vx in an xy-separator



$x,y \in V(G)$

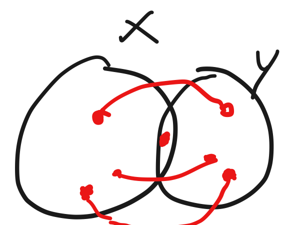
if $xy \in E(G)$ then $k(x,y) = k_H(x,y) + 1$

$H = G - xy$.

$k(G) = \min(k(x,y) : x,y \in G,\ x \neq y)$.

<u>Menger</u>

Two sets X, Y



strict X-Y path has only first vx in X only last in Y.

Z detaches X from Y if $G - Z$ has no X-Y paths.

<u>Thm 2.1</u> Given $X, Y \subseteq V(G)$ let k be the min nr of elements in a detaching set. Then $\exists$ a set of k disjoint strict X-Y paths.

<u>Thm 2.2</u> $x,y \in V(G)$ The maximum nr. of disjoint (other than at x and y) xy-paths is $k(x,y)$.

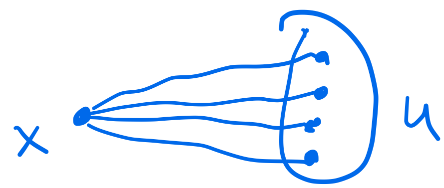
$k(G) \geq k$.

<u>Thm 2.5</u> G is k-connected

$\iff$ for any 2 vxs in G there are k disjoint paths joining them.

Thm 2.6

$\iff$ for any k-set U and vx $x \in V \setminus U$ we have an x-U fan.



Minimally k-connected graphs

G is k-connected, but no $G - xy$ is for any $xy \in E$.

If $k=1$ what are the minimally k-connected graphs? Trees.

If every edge is incident to a vx of degree k then G must be minimally k-connected.
Converse is not true.

Thm 4.3 G is minimally k-connected

$\implies \delta(G) = k$.

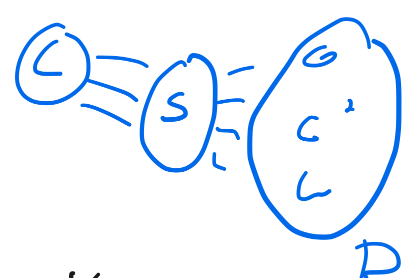
Lemma 4.1 G minimally k-connected, $xy \in E$,

S is a $(k-1)$-separator of $G_{xy} = G - xy$.
Then G_{xy} has 2 components, one with x and one with y.

<u>Lemma 4.2</u> k-connected graph G is minimally k-connected $\iff$ $\kappa(x,y) = k$ for every $xy \in E$.

<u>Proof</u> If $G = K_{k+1}$ then done.
Suppose $|G| \geq k+2$. Then G has separating k-sets. Let S be a separating k-set for which one of the components C has minimum order.

$D = G - S - C \qquad |D| \geq |C|$.



Goal: show that C is a single vx.

Suppose $|C| \geq 2$. Then C has an edge xy.
Let T be a $(k-1)$-separator in $G_{xy} = G - xy$.
$\hookrightarrow$ separates x from y.

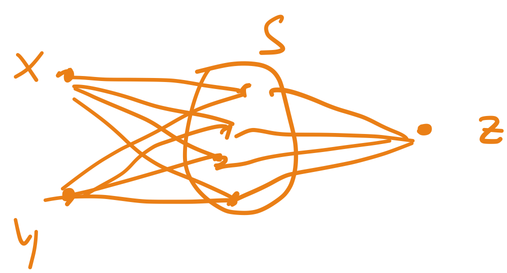
$\mathcal{G}_{xy}$ has a x-S fan. F_x

Otherwise we can find a separator by replacing a vx in S by y that leaves a smaller C.

Also an y-S fan. F_y.

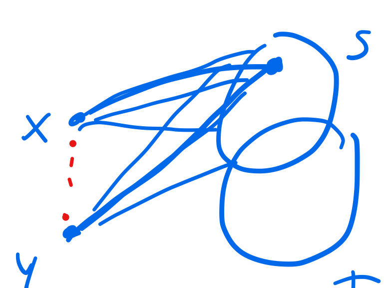
Now we claim that $D \subseteq T$. Suppose $z \in D \setminus T$. Then we have a z-S fan F_z in $\mathcal{G}_{xy}$.

Now T ($|T| = k-1$) does not separate x from z since we have k disjoint x-z paths. Also, T does not separate y from z. $\Rightarrow$ implies that T does not separate x from y. $\Rightarrow D \subseteq T$.



Let $r = |S \cap T|$.

For every vx u in $S \setminus T$, T must have some vx on



the x–u path or y–u path in F_x F_y respectively.

$\Rightarrow$ T has at least $\frac{1}{2}(k-r)$ vxs in C. Then D has at most

$$\underbrace{k-1}_{|T|} - \underbrace{r}_{\text{in } S} - \underbrace{\tfrac{1}{2}(k-r)}_{\text{in } C} = \tfrac{1}{2}(k-r) - 1.$$

vertices.

$|D| < |C|$ which is a contradiction. $\square$