These notes are mostly based on talks and notes by Steve Butler
(e.g. `https://www.d.umn.edu/~dfroncek/MCCCC_2018/abstracts/Butler_slides.pdf`, or
"Algebraic aspects of the normalized Laplacian").

Normalized Laplacian Matrices

The normalized Laplacian matrix is not as well-studied as the traditional Laplacian, but it combines some of the useful aspects of adjacency and Laplacian matrices.
The normalized Laplacian is defined as follows:

$$\mathcal{L} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}},$$

where D is the diagonal degree matrix as before. Note that we must assume there are no isolated vertices. (Those are uninteresting from a spectral perspective anyway.)
First, we look at ways in which the normalized Laplacian is related to the adjacency matrix. Have that when $A\vec{x} = \lambda\vec{x}$, then for every vertex v we have

$$\sum_{u,\ uv \in E(G)} x_u = \lambda x_v.$$

Now, suppose that $\mathcal{L}\vec{x} = \lambda\vec{x}$. We let $\vec{y} = D^{-\frac{1}{2}}\vec{x}$. We call this vector a *harmonic eigenvector*, and note that it is not an eigenvector of $\mathcal{L}$ although it is related to one.

Exercise 1. *Show that we have* 2 points

$$\sum_{u,\ uv \in E(G)} y_u = (1-\lambda) d_v y_v.$$

Exercise 2. *Show that the multiplicity of the eigenvalue 0 of A is equal to the multiplicity of the eigenvalue 1 of $\mathcal{L}$.* 1 point

Exercise 3. *Find an "easy" harmonic eigenvector and corresponding eigenvector of $\mathcal{L}$, and show that the multiplicity of the eigenvalue of 0 in $\mathcal{L}$ is equal to the number of connected components of G.* 1 point

Exercise 4. *Consider weighted graphs. Let cG be the graph G with all edgeweights multiplied by a (positive) constant c. Then $cA_G = A_{cG}$. Show that $\mathcal{L}_G = \mathcal{L}_{cG}$.* 1 point

As before, consider an equitable partition of a graph G into sets $V = V_1 \cup \cdots \cup V_k$. Recall that this means that for any $1 \leq i, j \leq k$ we have that every vertex in V_i has the same number of neighbors in V_j. Instead of degrees, we let

$$b_{ij} = |\{uv \ s.t.\ u \in V_i, v \in V_j\}|.$$

Note that edges within a set V_i are double-counted. Let $\mathcal{L}_B$ be the normalized Laplacian of the edge-weighted graph associated to this B matrix. Note that this graph is undirected. Then, we have

$$spec(\mathcal{L}_B) \subseteq spec(\mathcal{L}_G).$$

This is a result due to Butler and Chung (Handboook of Linear Algebra).

Exercise 5. *Use the previous results to find the normalized Laplacian spectrum of the path graphs P_n.* 3 points

Now, we will compare the normalized Laplacian to the traditional Laplacian. In particular, let's look at the Rayleigh quotient.

Exercise 6. *Show that* 1 points

$$\frac{\vec{x}^T \mathcal{L} \vec{x}}{\vec{x}^T \vec{x}} = \frac{\sum_{uv \in E(G)} (y_u - y_v)^2}{\sum_v d_v y_v^2}.$$

Exercise 7. *Show that* 2 points

$$0 \leq \frac{\vec{x}^T \mathcal{L} \vec{x}}{\vec{x}^T \vec{x}} \leq 2.$$

What do the multiplicities of the eigenvalues 0 and 2 tell us about G?

The following table summarized some aspects of graphs that can be deduced from te spectra of various associated matrices.

	nr. components	bipartite?	nr. bip components	nr. edges
A	no	yes	no	yes
L	yes	no	no	yes
Q	no	no	yes	yes
$\mathcal{L}$	yes	yes	yes	no