Signless Laplacian Matrix

We discuss one more version of the Laplacian matrix of a graph G: the signless Laplacian. This matrix is defined as

$$Q(G) = D(G) + A(G) = L(G) - 2A(G).$$

Exercise 1. *Show that if the diameter of a graph G is d, then $Q(G)$ has at least $d+1$ eigenvalues. (As we already know about $A(G)$ and $L(G)$.)* 1 point

Exercise 2. *We had that* 1 point

$$\vec{x}^T L \vec{x} = \sum_{uv \in E(G)} (x_u - x_v)^2.$$

Write a similar expression for the signless Laplacian.

Exercise 3. *What can you say about the multiplicity of the eigenvalue 0?* 2 points

Let $S(G)$ be the signless version of the edge incidence matrix: i.e. an $n \times m$ matrix, which has a 1 in row v and column e if $v \in e$.

Exercise 4. *Show that $Q = SS^T$.* 1 point

Exercise 5. *Show that for any $n \times m$ matrix A and $m \times n$ matrix B, we have* 4 points

$$\lambda^n \det(\lambda I_n - AB) = \lambda^m \det(\lambda I_m - BA),$$

and conclude that AB and BA have the same spectrum (including multiplicities), except for the 0 eigenvalue.

Exercise 6. *Let $Line(G)$ be the line graph of G. Show that all eigenvalues of $A(Line(G))$ are bounded from below by -2.* 3 points

Exercise 7. *Use the facts from the previous questions to explain the relationship between the spectra of $A(Line(G))$ and $Q(G)$.* 3 points

Transition matrices

A (linear) *discrete dynamical system* takes the form

$$\vec{x}(t) = P\vec{x}(t-1), \text{ with some initial condition } \vec{x}(0) = \vec{x}_0.$$

We can write this as a direct formula

$$\vec{x}(t) = P^t \vec{x}_0.$$

We define a *Markov chain* on (finitely many) states $S = \{1, 2, \ldots, n\}$ as a process that starts in some state and moves to another state (or stays put) at each discrete time step. So, at each time step t, we have a random variable X_t which represents the state of the process at time t. A *Markov process* is time-homogeneous and memoryless. This means that the random variables $X_1, X_2, \ldots$ are identically distributed, and that at each time step, the probability distribution of the next step of the system depends only on the current state, i.e.

$$\mathbb{P}(X_{t+1} = k | X_1 = x_1, \ldots, X_t = x_t) = \mathbb{P}(X_{t+1} = k | X_t = x_t).$$

We call these the transition probabilities:

$$p_{ji} = \mathbb{P}(X_{t+1} = j | X_t = i),$$

which we can represent in a $n \times n$ *transition matrix* P.

We call $\vec{x}$ a *distribution vector* if it has nonnegative elements that add up to 1. We then have the following result.

Exercise 8. *Show if P is a transition matrix and $\vec{x}$ is a distribution vector, then $P\vec{x}$ is a distribution vector. In fact, the transformation P always preserves the sum of elements of a vector.* 0 points

Exercise 9. *Show that if $\vec{x}_t$ represents the distribution of probabilities of X_t taking values in $\{1, 2, \ldots, n\}$, then $\vec{x}_{t+1} = P\vec{x}_t$ represents the distribution of probabilities of X_{t+1} taking values in $\{1, 2, \ldots, n\}$.* 1 point

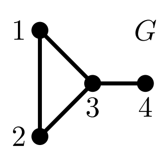
Exercise 10. *Show that if If P is a primitive $n \times n$ transition matrix, then P has exactly one distribution eigenvector $\vec{x}$ with eigenvalue 1 (where 1 is the largest eigenvector), meaning that $P\vec{x} = \vec{x}$. This is called the equilibrium distribution of P, and denoted by $\vec{x}_{equ}$.* 3 points
For any starting distribution $\vec{x}_0$, we have

$$\lim_{t \to \infty} P^t \vec{x}_0 = \vec{x}_{equ}.$$

Exercise 11. *Show that if P has a unique equilibrium, this does not imply that P is primitive.* 1 point

Number of returns in a random walk on a finite graph

The following example and proof are taken from Bollobás' Modern Graph Theory. We've spent some time thinking about Markov chains on a finite set of states, which can be thought of as random walks on a finite graph. Now we'll look in more detail at the very basic case where a walker is on a finite, simple, undirected graph $G(V, E)$. The walker starts at a starting vertex $v_0 \in V(G)$, and at each time step, they move from their current vertex v to a neighboring vertex of v, choosing one uniformly from the set $\Gamma(v)$ (the neighborhood of v). For example, the following graph G has the transition matrix M:



$$M = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 0 \end{pmatrix}$$

Exercise 12. *Write P of this random walk in terms of $A(G)$ and $D(G)$.* 0 points

Exercise 13. *Show that when G is a connected and non-bipartite graph, then* 2 points

$$\vec{x} = \frac{1}{2m}\begin{pmatrix} d(1) \\ d(2) \\ \vdots \\ d(n) \end{pmatrix},$$

where $m = |E(G)|$ is the equilibrium vector of the random walk on G.