Laplacian Matrix

We now introduce the Laplacian matrix of a graph, which is the more commonly used matrix when analyzing spectra of graphs. Over the next few weeks you will be convinced of its usefulness. We define the Lapcian as follows (there are related, but differing definitions in the literature). Let L be the Laplacian of an undirected unweighted graph G. Then

$$L_{uv} = \begin{cases} d(u), & u = v \\ -1, & uv \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 1. *If G is d-regular, what is the relationship between its (adjacency) spectrum and Laplace spectrum?* 1 point

Exercise 2. *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, and $\mu_1, \ldots, \mu_n$ be the eigenvalues of L. Show that* 1 point

$$2|E(G)| = \sum_i \lambda_i^2 = \sum_i \mu_i.$$

Exercise 3. *Find the Laplace eigenvalues of the path graph P_n.* 2 point

Exercise 4. *Describe the value $(L\vec{x})_u$ in terms of values x_u and x_v of neighbors of u.* 0 point

Exercise 5. *How would you interpret the number $\vec{x}^T L\vec{x}$?* 0 point

It is easy to see that the all-ones vector is an eigenvector of L for any graph; not just regular graphs. In this case, however, it turns out the have the smallest eigenvalue rather than largest.

Exercise 6. *Find the multiplicity of the eigenvalue 0 for a connected graph. What happens with disconnected graphs?* 1 point

Note that Figure 1.1 on p.7 in Brouwer-Haemers shows that the adjacency spectrum does not help us distinguish between connected and disconnected graphs.

Theorem 1. *The Laplacian of an undirected unweighted graph has nonnegative eigenvalues.*

This follows directly from the following fact about matrices. Let M be an $n \times n$ complex matrix. Let $r_i(M) = \sum_{j \neq i} M_{ij}$. Then a Gershgorin disc is the ball (disc) $B(m_{ii}, r_i(M))$ in the complex plane.

Theorem 2 (Gershgorin's Circle Theorem)**.** *All eigenvalues of M lie in $\bigcup_i B(m_{ii}, r_i(M))$.*

We used Theorem 2.2.1 (p.24) in Spielman to prove parts of Perron-Frobenius. This theorem states that the Rayleigh quotients of vectors are bounded between the smallest and largest eigenvalues of a real symmetric matrix. Let μ_{min} and μ_{max} be the smallest and largest eigenvalues of L, respectively. For any vector nonzero $\vec{x}$

$$\mu_{min} \leq \frac{\vec{x}^T L\vec{x}}{\vec{x}^T\vec{x}} \leq \mu_{max},$$

and these extrema are achieved exactly by the eigenvectors of μ_{min} and μ_{max}, respectively.

Exercise 7. *Use the above fact to show that all eigenvalues of L are nonnegative.* 2 points