Exercise 1. *Find the spectrum of the hypercube graphs Q_n.* 1 point

Consider a *locally bijective homomorphism* from a graph G to a graph H. This is a map $\psi : V(G) \to V(H)$ such that for each vertex $v \in V(G)$ the map restricted to $N(v)$ is a bijection to $N(\pi(v))$.

Exercise 2. *Show if there exists a locally bijective homomorphism from G to H, then the spectrum of H is a subset of the spectrum of G.* 1 point

Consider a "double cover" of a graph H. This is a graph G on twice the number of vertices as H and a 2-1 locally bijective homomorphism. Then consider a matrix B, obtained by signing the adjacency matrix A_H depending on whether edges in G are switched or not.

Exercise 3. *Show that the spectrum of the matrix B is a subset of the spectrum of A_G.* 3 point

Exercise 4. *Show that Q_3 is a double cover of K_4 and use the above two exercises to find its spectrum.* 2 point

Seidel switching

We define the *Seidel adjacency matrix S* of a graph G as follows:

$$S_{uv} = \begin{cases} 0, & u = v, \\ -1, & uv \in E(G), \\ 1, & uv \notin E(G). \end{cases}$$

Exercise 5. *Let G be a k-regular graph. How does the spectrum of S relate to the spectrum of G?* 1 point

A *Seidel switch* is obtained by complementing the edges across a cut of G. What is the result on the Seidel matrix? Seidel switching gives us an equivalence relation on graphs.

Exercise 6. *How do the Seidel adjacency spectra of two graphs relate if they are in the same switching class?* 1 point

Godsil-McKay switching

Let G be a graph with a vertex partition $V(G) = V_1 \cup \cdots \cup V_k \cup D$, such that the V_is induce an equitable partition, and every vertex $v \in D$ has the property that for each V_i, then the number of neigbors of v in V_i is either 0, $\frac{1}{2}|V_i|$ or $|V_i|$. A Godsil-McKay switch creates a new graph from G by taking each pair consisting of a vertex $v \in D$ and V_i such that v has $\frac{1}{2}|V_i|$ neighbors in V_i, and switches those to the other $\frac{1}{2}|V_i|$. Note that this does not change the degree of any $v \in D$, but it might change the degrees of other vertices.

Exercise 7. *Show that two graphs related by a Godsil-McKay switch are cospectral.* 3 points

Perron-Frobenius

We say that a nonnegative $n \times n$ matrix A is *primitive* if there is a t such that A^t is positive. We say that A is *irreducible* if for every $1 \leq i, j \leq n$ there is a t such that $(A^t)_{ij} > 0$.

Exercise 8. *Show that primitive and irreducible are not equivalent definitions, and explain what they mean in terms of adjacency matrices of (directed/undirected) graphs.* 1 point

Exercise 9. *Show that if T is irreducible, then $I + T$ is primitive.* 1 point

The *period* of an irreducible matrix is the greatest common divisor of all the values t such that $(A^t)_{ii} > 0$, for some $1 \leq i \leq n$.

Exercise 10. *Show that the definition of the period is indeed independent of the i chosen.* 1 point

We let ρ be the *spectral radius* of A, defined as $\rho = \max_i |\lambda_i|$ taken over all eigenvalues of A.

Theorem 1. *Let $A \geq 0$ be irreducible. Then the following hold.*

(i) A has a unique positive real eigenvalue $\lambda_1 = \rho$, with algebraic and geometric multiplicity 1, and which has a positive eigenvector.

(ii) If A is primitive, then $|\lambda_i| = \rho$ implies that $\lambda_i = \lambda_1$. Otherwise, if A has period d, then there are d eigenvalues with absolute value ρ, and they are precisely the values $\rho e^{2\pi i k/d}$ for $k = 0, 1, \ldots, d-1$. In fact, the spectrum of A is invariant under a rotation over $2\pi/d$ of the complex plane.