# Paths, products and equitable partitions

We will continue to work this week with the adjacency matrices of simple graphs. From last week (and a little of this week), we know the following spectra for a few standard graph classes:

$$\begin{aligned} K_n &: \; n-1, -1, -1, \ldots, -1 \\ K_{n_1,n_2} &: \; \sqrt{n_1 n_2}, -\sqrt{n_1 n_2}, 0, \ldots, 0 \\ C_n &: \; 2\cos\left(\frac{2\pi k}{n}\right), \; 0 \le k \le n-1, \\ P_n &: \; 2\cos\left(\frac{\pi k}{n+1}\right), \; 1 \le k \le n. \end{aligned}$$

**Exercise 1.** *Find the spectrum of $P_n$, the path graph on $n$ vertices.* 3 points

**Exercise 2.** *Let $G$ be a $d$-regular graph. Investigate the eigenvalue spectrum of $\overline{G}$. Use this method to find the spectrum of $K_{n,n}$.* 1 point

## Graph products

We define three useful graph products. The *tensor* or *Kronecker product* $G \times H$, the *Cartesian product* $G \Box H$, and the strong product $G \boxtimes H$. They are defined as follows:

$$\begin{aligned} &V(G \times H) = V(G \Box H) = V(G \boxtimes H) = V(G) \times V(H), \\ &E(G \times H) = \{(u,v)(x,y) \mid ux \in E(G) \text{ and } vy \in E(H)\}, \\ &E(G \Box H) = \{(u,v)(x,y) \mid u = x \text{ and } vy \in E(H) \text{ or } ux \in E(G) \text{ and } v = y\}, \\ &E(G \boxtimes H) = E(G \times H) \cup E(G \Box H). \end{aligned}$$

**Exercise 3.** *For two graphs $H$ and $G$, write the adjacency matrix $M_{G \times H}$ in terms of $M_G$ and $M_H$, and find the spectrum of $G \times H$ in terms of the spectra of $G$ and $H$.* 1 point

**Exercise 4.** *For two graphs $H$ and $G$, write the adjacency matrix $M_{G \Box H}$ in terms of $M_G$ and $M_H$, and find the spectrum of $G \Box H$ in terms of the spectra of $G$ and $H$.* 1 point

**Exercise 5.** *For two graphs $H$ and $G$, write the adjacency matrix $M_{G \boxtimes H}$ in terms of $M_G$ and $M_H$, and find the spectrum of $G \boxtimes H$ in terms of the spectra of $G$ and $H$.* 1 point

## Equitable partitions

An *equitable partition* of a graph $G$ is a partition of the vertices $V = V_1 \cup \cdots \cup V_2$, such that there exist constants $d_{ij}$ for $1 \le i, j, \le n$ such that each vertex in $V_i$ has exactly $d_{ij}$ neighbors in $V_j$.

**Exercise 6.** *Find a nontrivial equitable partition of a graph for which you know the eigenvalue spectrum, and look at the eigenvalues of the matrix $B$ with $B_{ij} = d_{ij}$. Formulate a conjecture.* 1 point

**Exercise 7.** *Prove your conjecture from the previous exercise.* 2 point