Minimum rank over $\mathbb{F}_2$

We start with a warm-up exercise.

Exercise 1. *Show that the rank over $\mathbb{F}_2$ of the adjacency matrix of a graph is always even.* 2 points

For the last week of class, we will do a little bit of linear algebra over $\mathbb{F}_2$. A *faithful orthogonal representation* over $\mathbb{F}_2$ of a graph is an assignment of vectors in $\vec{u}_i \in \mathbb{F}_2^k$, for $1 \le i \le n$ to the vertices of G, such that $\vec{u}_i \cdot \vec{u}_j = 0$ for $i \neq j$ if and only if $ij \notin E(G)$. We denote by $c_2(G)$ the minimum k for which such a representation exists. Let $\mathrm{mr}_2(G)$ be the minimum rank over all matrices that fit G over $\mathbb{F}_2$ (where the rank is also taken over $\mathbb{F}_2$).

Exercise 2. *Show (in two ways) that* 2 points

$$\mathrm{mr}_2(G) \le c_2(G).$$

In fact, these two invariants are closely related. In class, we outline the proof for the following theorem. This and more can be found in the following paper: `https://www.combinatorics.org/ojs/index.php/eljc/article/view/v29i1p38`

Theorem 1. *For any graph G, we have*

$$\mathrm{mr}_2(G) \le c_2(G) \le \mathrm{mr}_2(G) + 1.$$

For the n-dimensional hypercube Q_n, we have that $\mathrm{mr}_2(Q_n) = 2^{n-1}$. Last week, we briefly discussed a proof described in "Zero forcing sets and the minimum rank of graphs" (AIM Minimum Rank – Special Graphs Work Group), which is based on a recursive construction of a minimum rank matrix that fits Q_n. We now give an alternative proof using faithful orthogonal representations.

Exercise 3. *Show that $\mathrm{mr}_2(Q_n) \ge 2^{n-1}$. Do not use zero-forcing. Instead, look at the adjacency matrix of Q_n directly.* 2 points

Theorem 2. *For the n-dimensional hypercube Q_n, we have*

$$\mathrm{mr}_2(Q_n) = 2^{n-1}.$$

Proof. We only need to show that $\mathrm{mr}_2(Q_n) \le 2^{n-1}$, and we do that by showing that $c_2(Q_n) = 2^{n-1}$. Consider the following representation of the vertices of Q_n. Let S be the set of vertices in $V(Q_n)$ such that their bitstrings have an even number of 1s. Note that this is a maximum independent set. Let $N[k] = N(k) \cup \{k\}$. For each $k \in S$, and for $1 \le i \le n$, let

$$(\vec{v}_k)_i = \begin{cases} 1, & \text{if } i \in N[k], \\ 0, & \text{otherwise.} \end{cases}$$

Then, we let X be the matrix whose columns are $\vec{v}_1, \ldots, \vec{v}_{2^{n-1}}$. We claim that the rows of X are a faithful orthogonal representation of Q_n and therefore XX^T is a matrix that fits the adjacency matrix of Q_n.

Exercise 4. *Prove the above claim.* 2 points

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