Minimum rank and zero forcing

Let G be a simple graph. We will consider a family of matrices that contain the information in G, and look for the smallest rank among them. In a sense, this measures the amount of information in G. Formally, we say that an $n \times n$ matrix A fits a graph G if, for any pair $i \neq j$, we have that $A_{ij} = 0$ if and only if $ij \notin E(G)$. Note that A can have any values on the diagonal. Then, we let the *minimum rank* of G be defined as

$$\operatorname{mr}(G) = \min\{\operatorname{rank}(A) \mid A \text{ fits } G\}.$$

Exercise 1. *Find the minimum rank of K_n and $K_{p,q}$.* 2 points

Exercise 2. *Show that* $\operatorname{mr}(G) \leq n-1$ *for any graph G.* 1 point

It turns out that $\operatorname{mr}(G) = n-1$ if and only if $G \sim P_n$.

Exercise 3. *Show that* $\operatorname{mr}(G) > \operatorname{diam}(G)$ *for any graph G.* 1 point

Exercise 4. *Show that for $v \in V(G)$,* 2 points

$$0 \leq \operatorname{mr}(G) - \operatorname{mr}(G - v) \leq 2,$$

and that for $e \in E(G)$

$$-1 \leq \operatorname{mr}(G) - \operatorname{mr}(G - e) \leq 1.$$

Consider the following game on a simple graph G. We start with a set $Z \subseteq V(G)$ and let $F = Z$ ("filled"). If there is a vertex $v \in V \setminus F$ such that v has exactly one neighbor u that is not in F, we add u to F. We continue until there are no such vertices in the graph. Let $Z(G)$ be the cardinality of a smallest set Z such that $F = V$. This is called the *zero forcing number*, and it has a surprising connection to the minimum rank problem.

Theorem 1. *For any n-vertex graph G, we have* $n - \operatorname{mr}(G) \leq Z(G)$.

Proof. We'll follow the proof in "Zero forcing sets and the minimum rank of graphs" by the AIM Minimum Rank – Special Graphs Work Group.
Let A be a matrix that fits G.

Exercise 5. *Show that if* $\operatorname{null}(A) > k$ *then for any $S \subseteq V$ with $|S| = k$, there exists a nonzero $\vec{x} \in \ker(A)$ that is 0 at every index in S.* 1 point

Exercise 6. *Show that if Z is a zero forcing set of G and $\vec{x} \in \ker(A)$ such that $\vec{x}$ is 0 at every index in Z, then $\vec{x} = \vec{0}$. Then, complete the proof.* 1 point

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