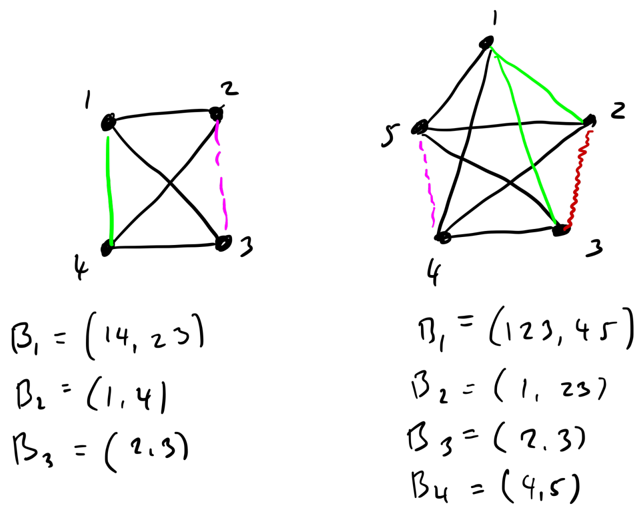
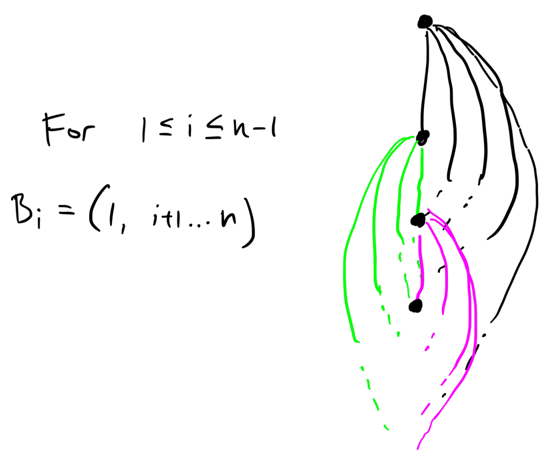
Graham-Pollak

We ask the following question: what is the smallest partition of the edges of K_n into complete bipartite graphs? Below are two examples of K_4 and K_5. We indicate the complete bipartite graphs B_i by their two partite vertex sets (X_i, Y_i).



In general, we can partition the edge set of any complete graph K_n into $n-1$ complete bipartite graphs. For example, by using the following general construction, in which each B_i is a star graph.



Let $\mathrm{bp}(G)$ indicate the smallest number of bipartite graphs needed to partition the edges of a graph G. We call this the *biclique partition number*. It turns out that for K_n, we can't do any better than the above star strategy.

Theorem 1 (Graham-Pollak)**.** *We have that* $\mathrm{bp}(K_n) = n - 1$.

Proof. The upper bound is shown in the example above, so we only need to show that $\mathrm{bp}(K_n) \geq n-1$. Let $B_1 = (X_1, Y_1), \ldots, B_k = (X_k, Y_k)$ be a biclique partition of K_n. For $1 \leq i \leq n$ let $\vec{x}_i$ and $\vec{y}_i$ be $n \times 1$ indicator vectors of the sets X_i and Y_i:

$$(\vec{x}_i)_v = \begin{cases} 1, & \text{if } v \in X_i, \\ 0, & \text{otherwise.} \end{cases}, \quad (\vec{y}_i)_v = \begin{cases} 1, & \text{if } v \in Y_i, \\ 0, & \text{otherwise.} \end{cases}.$$

Let

$$A_i = \vec{x}_i \vec{y}_i^T,$$

and let

$$A = \sum_{i=1}^{k} A_i.$$

Exercise 1. *Show that* 1 point

$$J - I = A + A^T.$$

Exercise 2. *Show that* 1 point

$$\text{rank}(A) \leq k.$$

Exercise 3. *Show that if* 1 point

$$\text{rank}(A) \leq n - 2,$$

then there exists a vector $\vec{v}$ that satisfies both $A\vec{v} = 0$ and $\mathbb{1}^T \vec{v} = 0$, and finish the proof.

□

Witsenhausen

We now show a more powerful theorem by Witsenhausen, of which Graham-Pollak is a corollary. For a simple graph G and its adjacency matrix A_G, let $n_+(A_G)$ be the number of (strictly) positive eigenvalues of its adjacency matrix, and $n_-(A_G)$ the number of (strictly) negative eigenvalues.

Theorem 2 (Witsenhausen)**.** *We have*

$$\text{bp}(G) \geq \max(n_+(A_G), n_-(A_G)).$$

Proof. We will show that $\text{bp}(G) \geq n_+(A_G)$, and the proof for $n_-(A_G)$ is almost identical. Let $B_1 = (X_1, Y_1), \ldots, B_k = (X_k, Y_k)$ be a biclique partition of K_n. For $1 \leq i \leq n$ let $\vec{x}_i$ and $\vec{y}_i$ be $n \times 1$ indicator vectors of the sets X_i and Y_i as before. Let

$$W = \{\vec{w} \in \mathbb{R}^n \mid \vec{w}^T \vec{x}_i = 0,\ \forall i \in \{1, \ldots, k\}\}.$$

We also let

$$P = \{\vec{p} \in \mathbb{R}^n \mid A_p \vec{p} = \lambda \vec{p},\ \lambda > 0\}.$$

Exercise 4. *Show that $W \cap P = \{\vec{0}\}$ and finish the proof.* Hint: consider $\vec{w}^T A_G \vec{w}$ for $\vec{w} \in W$ and $\vec{p}^T A_G \vec{p}$ for $\vec{p} \in P$. 2 points

□