Shannon capacity and Lovász number

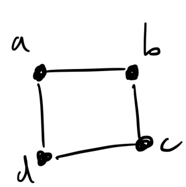
We are interested in the rate at which information can be transmitted through a noisy channel. We have a set of n words, some of which may be confused for one another due to noise/errors. This is represented by a *confusion graph* G. We want to find an optimal set of messages to send which cannot be confused for one another. If we send one word at a time, this implies we need a maximum independent set in the graph. If we send messages that include two words, then two messages xy and uv can be confused for one another if one of the following holds:

- $xu, yv \in E(G)$,
- $x = u$ and $yv \in E(G)$,
- $y = v$ and $xu \in E(G)$.

In other words, we look for a maximum independent set in $G \boxtimes G$. The *Shannon capacity* of a graph measures its efficiency in terms of the optimal set of messages:

$$\Theta(G) = \max_k \{\alpha(G^k)^{1/k}.\}$$

Below are examples with messages up to length 2. We see that $\Theta(C_4) \geq 2$ and $\Theta(C_5) \geq \sqrt{5}$.

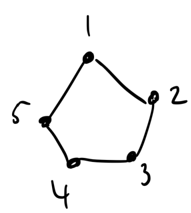


1 word

$\begin{cases} a \\ c \end{cases} \Rightarrow 2$

2 words

$\begin{cases} aa \\ bc \end{cases} \Rightarrow \sqrt{2}$



1 word

$\begin{cases} 1 \\ 3 \end{cases} \Rightarrow 2$

2 words

$\begin{cases} 11 \\ 23 \\ 45 \\ 35 \\ 24 \end{cases} \Rightarrow \sqrt{5}$

Theorem 1 (Shannon). *We have*

$$\alpha(G) \leq \Theta(G) \leq \chi(\overline{G}).$$

Theorem 2 (Shannon). *We have*

$$\Theta(G \boxtimes H) = \Theta(G)\Theta(H).$$

An *orthonormal representation* of a graph G is a set of vectors $U = \{\vec{u_1}, \ldots, \vec{u_n}$ such that

$$u_i^T u_j = \begin{cases} 1, & \text{if } i = j, \\ 0 & \text{if } ij \notin E(G). \end{cases}$$

The Lovász number of a graph is defined as

$$\theta(G) = \min_{U} \max_{i \in V(G)} \frac{1}{(\vec{e_1}^T \vec{u_i})^2},$$

where the minimum is taken over all orthonormal representations $U = \{\vec{u_1}, \ldots, \vec{u_n}$ of G.

Exercise 1. *Show that* 2 points

$$\alpha(G) \leq \theta(G) \leq \Theta(G).$$

Exercise 2. *Find* $\Theta(C_4)$. 1 point

Exercise 3. *Find* $\Theta(C_5)$. 3 points