Interlacing

As a warm-up to interlacing, let G be a graph and let A_G be its adjacency matrix. Then we can say the following about $G - v$, for any vertex $v \in V(G)$, and its adjacency matrix.

Exercise 1. *Show that if A_G has eigenvalue λ with multiplicity k, then A_{G-v}, for any $v \in V(G)$, has eigenvalue λ with multiplicity at least $k - 1$.* 1 point

In fact, we can say something more precisely, about all eigenvalues of G and $G - v$. Let A be a real, symmetric $n \times n$ matrix, and B be a real, symmetric $m \times m$ matrix with $m < n$. Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of A and $\mu_1 \leq \cdots \leq \mu_m$ be the eigenvalues of B. Then we say that the eigenvalues of A and B *interlace* if

$$\lambda_i \leq \mu_i \leq \lambda_{n-m+i}, \text{ for all } 1 \leq i \leq m.$$

Lemma 1. *Let A be a real, symmetric $n \times n$ matrix. Let S be a subset of $[n]$ with $|S| = m$, and let B be obtained from A by keeping the columns and rows with indices in S. Then the eigenvalues of A and B interlace.*

Proof. Without loss of generality, suppose that A has the form

$$A = \begin{pmatrix} B & X^T \\ X & C \end{pmatrix}.$$

Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of A and $\mu_1 \leq \cdots \leq \mu_m$ be the eigenvalues of B, with eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ and $\vec{w}_1, \ldots, \vec{w}_n$, respectively. We will show only that $\lambda_i \leq \mu_i$, and the other side will be very similar. Let

$$V = \operatorname{span}(\vec{v}_i, \ldots, \vec{v}_n)$$

and let

$$W = \operatorname{span}(\vec{w}_1, \ldots, \vec{w}_i).$$

We also let

$$\tilde{W} = \left\{ \begin{pmatrix} \vec{w} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n \mid \vec{w} \in W \right\}.$$

It is not hard to see that for any $\vec{w}$ and associated $\tilde{\vec{w}}$ we have that

$$\vec{w}^T A \vec{w} = \tilde{\vec{w}}^T B \tilde{\vec{w}}.$$

By counting dimensions, we see that there must exist some non-trivial $\tilde{\vec{w}} \in V \cap \tilde{W}$. By Courant-Fisher, we now have

$$\lambda_i \leq \frac{\vec{w}^T A \vec{w}}{\vec{w}^T \vec{w}} = \frac{\tilde{\vec{w}}^T B \tilde{\vec{w}}}{\tilde{\vec{w}}^T \tilde{\vec{w}}} \leq \mu_i.$$

□

Exercise 2. *Let S be a real $n \times m$ matrix such that $S^T S = I_m$. Let A be a real, symmetric $n \times n$ matrix, and let $B = S^T A S$. Show that A and B interlace.* 3 points

The above results tell us that A_G and A_{G-v} have interlacing eigenvalues, but we cannot directly apply this to the Laplacian matrices.

Exercise 3. *Show that the (signless) Laplacian matrices of G and $G-e$ respectively for any edge $e \in E(G)$ interlace.* 2 points

Exercise 4. *Show that the (signless) Laplacian matrices of G and $G-v$ respectively for any vertex $v \in V(G)$ interlace.* 1 points

Independent sets

This Section is from Brouwers and Haemers 3.5.
Let $\alpha(G)$ be the size of a largest independent set. As a warm-up exercise, note that if G has an independent set of size α, then it has an $\alpha \times \alpha$ matrix of all 0s as a principal submatrix.

Exercise 5. *Use interlacing to show that* 1 point

$$\alpha(G) \leq |\{i \,:\, \lambda_i \leq 0\}|$$

and

$$\alpha(G) \leq |\{i \,:\, \lambda_i \geq 0\}|.$$

Let $V = V_1 \cup \cdots \cup V_m$ be any partition of V. Order the vertices by these partition classes to obtain

$$A = \begin{pmatrix} A_{1,1} & \dots & A_{1,m} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,m} \end{pmatrix}.$$

We construct a matrix $\tilde{B}$ whose entries are the average row sums of $A_{i,j}$. This is a generalization of the B matrices we used with equitable partitions (when those row sums were constants). Let $\tilde{S}$ be the $n \times m$ matrix that has a 1 in position i,j if $i \in V_j$ and 0 otherwise. Then

$$(\tilde{B})_{i,j} = \frac{1}{|V_i|}(\tilde{S}^T A \tilde{S})_{i,j}.$$

Exercise 6. *Let* 1 point

$$D = \begin{pmatrix} |V_1| & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & |V_m| \end{pmatrix},$$

and let $B = D^{1/2}\tilde{B}D^{-1/2}$. Show that the eigenvalues of B interlace with those of A_G.

Exercise 7. *Show that* 1 point

$$\alpha(G) \leq n\frac{-\lambda_1\lambda_n}{\delta^2 - \lambda_1\lambda_n},$$

where δ is the minimum degree of G. Start with a partition $V = V_1 \cup V_2$ where V_1 is a largest independent set. Form the matrix B and consider its determinant.

Chromatic number

The chromatic number of a graph, denoted $\chi(G)$, is the smallest m such that there exists a partition $V = V_1 \cup \cdots \cup V_m$ such that all sets V_i are independent sets. We can find both upper and lower bounds on the chromatic number from eigenvalues of the adjacency matrix.

Theorem 2. *We have that* Prop 3.6.1

$$\chi(G) \leq 1 + \lambda_n,$$

with equality if and only if G is a complete graph or an odd cycle.

Theorem 3. *If G has at least one edge, then we have that* Thm 3.6.2

$$\chi(G) \geq 1 - \frac{\lambda_n}{\lambda_1}.$$

Exercise 8. *Prove Thm 3. Let $V = V_1 \cup \cdots \cup V_\chi$ be an optimal partition into independent sets. We would like to find a matrix B such that there exists some S so that $B = S^T AS$, but in this case we want λ_n to be an eigenvalue of both A and B. Let $\vec{x}$ be an eigenvector with eigenvalue λ_n. Let $\tilde{S}$ be the matrix that has the value x_i in position i, j if $i \in V_j$ and 0 otherwise.* 3 points