Introduction to graphs and matrices

Spectral graph theory lies in the intersection of linear algebra and graph theory. It is a very broad subject, with applications in different areas of research, such as graph theory (of course), physics, statistics, computer science, data science,... There are too many topics and applications to fit into one course, and we have flexibility regarding which to cover. So, please talk to me about your interests!

Graphs

A graph is a pair $G(V, E)$, where V is a set of *vertices* (nodes) and $E \subseteq \binom{V}{2}$ a set of *edges* (links). In this course, we will almost always consider graphs to be *simple*, meaning that they have no self-loops or multiple edges, and that edges are undirected. Let $d(v)$ indicate the *degree* of a vertex $v \in V(G)$, i.e. the number of edges that are *indicent* to it (contain v), or the number of vertices *adjacent* to v (that share an edge with v). A *path* is a graph of the form $P(V, E)$ with $V = \{v_0, \ldots, v_k\}$ and $E = \{v_0v_1, \ldots, v_{k-1}v_k\}$. A *cycle* is a graph of the form $C(V, E)$ with $V = \{v_0, \ldots, v_k\}$ and $E = \{v_0v_1, \ldots, v_{k-1}v_k, v_1v_k\}$. We say that a graph is *connected* if there exists a path in the graph between any pair of vertices.
We can store/communicate the information contained in a graph by, for example, drawing the graph, or listing the sets V and E, or maybe as a dictionary that keeps track of the set of neighbors of each vertex,... Here, however, we will focus on ways to store graphs as a matrix. As it turns out there are many ways to do this, each with their own advantages and applications.

Incidence matrix

We start with an *incidence matrix* A, which has a row for each vertex, and a column for each edge of G. We let $A_{ve} = 1$ if $v \in e$ and $A_{ve} = 0$ otherwise.
A famous result in graph theory is the so-called Handshake Lemma.

Lemma 1 (Handshake Lemma.)**.** *For any graph G, we have*

$$2|E| = \sum_{v \in V} d(v).$$

Exercise 1. *Use the incidence matrix of a graph G to prove the Handshake Lemma.* 1 point

Exercise 2. *As a variation on the incidence matrix, consider the following. For each edge $e = uv$ and associated column in the matrix A, instead of putting a 1 in both positions u and v, set one of them to 1 and one to -1 (this choice is arbitrary, so this matrix is not uniquely defined). Suppose that a graph G is connected, and consider such an associated matrix A. Can you describe cycles, forests and spanning trees in G in terms of linear algebra on the columns of A?* 2 points

Adjacency matrix

The *adjacency matrix* M (or often denoted as A) has a row for each vertex and a column for each vertex of G. We let $M_{uv} = 1$ if $uv \in E$ and $M_{uv} = 0$ otherwise.
Since we assume that graphs are simple, the adjacency matrix has only real-valued entries and is symmetric (and Hermitian), i.e. we have $M = M^T$. We will assume a few facts about symmetric matrices. You may try to prove these as an exercise.

Theorem 2. *Let M be a real, symmetric matrix. Then,*

(i) for any eigenvalue μ of M, its algebraic multiplicity is equal to its geometric multiplicity;

(ii) all eigenvalues of are real;

(iii) if $\vec{x}$ and $\vec{y}$ are eigenvectors of M with distinct eigenvalues λ and μ, respectively, then $\vec{x} \perp \vec{y}$. Therefore, there exists an orthonormal eigenbasis for the columnspan of M.

Exercise 3. *Show that if two graphs G_1 and G_2 are isomorphic (the same under reordering of the vertices) then their adjacency matrices M_1 and M_2 are similar, i.e. find a matrix B such that $B^{-1}M_1B = M_2$.* 1 point

Exercise 4. *Let G and H be two graphs. What can you say about the spectrum of the disjoint union $G \cup H$?* 1/2 point

Exercise 5. *Suppose that G is a k-regular graph. Give one eigenvector and associated eigenvalue of M.* 1/2 point

Exercise 6. *Show that the eigenvalues of G are bounded by $\Delta(G)$ (the maximum degree of G).* 1/2 point

Exercise 7. *Think of M as an operator on the set of functions $g : V(G) \to \mathbb{R}$ (write g as a column vector of length n such that $g(v) = \vec{g}_v$). For a vertex v, what is $(M\vec{g})_v$?* 1/2 point

Exercise 8. *What does the matrix M tell us about walks in the graph? Write the following invariants in terms of the matrix M: $d(u,v)$ (distance between two vertices u and v), mboxecc(v) (the eccentricity of a vertex v: maximum $d(u,v)$ over all $u \in V$) and diam(G) (the diameter of the graph: maximum $d(u,v)$ over all $u, v \in V$).* 2 points

Exercise 9. *One more fact about symmetric matrices is that if $\mu_1, \ldots, \mu_k$ are the unique eigenvalues of M, then $(M-\mu_1 I)(M-\mu_2 I)\ldots(M-\mu_k I) = 0$ (this is the minimal polynomial of M). Use this to show that M^l for any $l \geq 0$ can be written as a linear combination of $I, M, M^2, \ldots, M^{k-1}$. Then, conclude that the number of unique eigenvalues of M must exceed the diameter of G. What does this tell us about the spectrum of the path graph P_n?* 2 points

Exercise 10. *Find the eigenvalues of the adjacency matrix of K_n, the complete graph on n vertices.* 1 point

Exercise 11. *Find the eigenvalues of the adjacency matrix of K_{n_1,n_2}, the complete bipartite graph with partite sets of order n_1 and n_2 respectively.* 2 points

Exercise 12. *Find the eigenvalues of the adjacency matrix of C_n, the cycle graph on n vertices. Start with a directed cycle (not a symmetric matrix).* 3 points