Our next goal in the analysis of $G(n, p)$ is to describe the evolution of the sizes of the connected components. Once these get larger than constant size, the tools we have been using so far fall short, we first need a few additional tools.

Poisson Distribution

Definition 1. *A random variable X has* Poisson distribution *with mean λ if*

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!},$$

for $k = 0, 1, 2, \ldots$

The next exercise makes it clear why we call the parameter λ the mean of the distribution.

Exercise 1. *Show that if X has Poisson distribution with mean λ, then $\mathbb{E}(X) = \mathrm{var}(X) = \lambda$.* 2

Poisson distributions have several properties that make them easier to work with than binomial distributions, and in the case where a binomial distribution (with parameters n and p) has $np \ll n$, they are a good approximation to the binomial. To be more precise, suppose that Y has a binomial distribution. Let $n \to \infty$, and suppose that $p = \frac{\lambda}{n}$, where λ is a constant. This gives us a constant expected value $\mathbb{E}(Y) = \lambda$. We'll show that in the limit of n, Y has a Poisson distribution with mean λ. Recall from the first set of notes that

$$\left(1 - \frac{a}{n}\right)^n \to e^{-a}, \quad \text{as } n \to \infty.$$

We have, for any constant value k,

$$\begin{aligned}
\mathbb{P}(Y = k) &= \binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} \frac{n!}{(n-k)! n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&\to \frac{\lambda^k e^{-\lambda}}{k!}, \quad \text{as } n \to \infty,
\end{aligned}$$

since we see that

$$\frac{n!}{(n-k)! n^k} = \frac{n(n-1)\ldots(n-k+1)}{n^k} \to 1, \quad \text{as } n \to \infty.$$

Probability generating functions

Generating functions form a way of encoding sequences and provide a new set of tools to study and/or manipulate them. They have many powerful applications and if you haven't seem them before, I recommend that you read Chapter 10 in Invitation to Discrete Mathematics by Matoušek and Nešetřil.

Definition 2. *The (ordinary) generating function of a sequence* $\{a_n\}_{n=0}^{\infty}$ *is defined as*

$$f_a(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \dots$$

Exercise 2. *For two sequences* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$*, we define their* convolution $\{c_n\}_{n=0}^{\infty}$ *as the sequence defined by* 1

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

Show that if $\{a_n\}_{n=0}^{\infty}$ *has generating function* $f_a(s)$ *and* $\{b_n\}_{n=0}^{\infty}$ *has generating function* $f_b(s)$*, then the sequence* $\{c_n\}_{n=0}^{\infty}$ *has generating function* $f_c(s) = f_a(s) \cdot f_b(s)$.

Definition 3. *Let* X *be a nonnegative integer-valued random variable, with probability distribution given by* $\mathbb{P}(X = 0) = p_0$, $\mathbb{P}(X = 1) = p_1, \dots$ *Then we define the probability generating function of* X *as*

$$f_X(s) = \mathbb{E}(s^X) = p_0 + p_1 s + p_2 s^2 + p_3 s^3 + \dots$$

We will not prove this fact here, but probability generating functions are unique: if two probability distributions have the same probability generating function then they are the same distribution.

Exercise 3. *Show that* $f_X(0) = p_0$ *and* $f_X(1) = 1$. 1

Exercise 4. *Show that* $f'_X(1) = \mathbb{E}(X)$. 1

Exercise 5. *Similarly to the previous exercise, can you derive* $\mathbb{E}(X^2)$ *from* $f_X(s)$*?* 2

Exercise 6. *Show that if two random variables* X *and* Y *are independent, then* 1

$$f_{X+Y}(s) = f_X(s) \cdot f_Y(s).$$

Exercise 7. *Use probability generating functions to show that if* X *is a Poisson random variable with mean* λ_X *and* Y *is a Poisson random variable with mean* λ_Y*, and* X *and* Y *are independent, then* $X + Y$ *is a Poisson random variable with mean* $\lambda_X + \lambda_Y$. 2

Probability generating functions give us a second way to show the Poisson approximation to the binomial. Let X be a binomial random variable with parameters n and p. First of all, we see that, for the general binomial distribution, we have

$$f_X(s) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} s^k = (1 - p + ps)^n.$$

Now, let $p = \lambda/n$. Then we have

$$f_X(s) = (1 - p + ps)^n = \left(1 - \frac{\lambda(1-s)}{n}\right)^n \to e^{-\lambda(1-s)} = f_Y(s),$$

if Y is a Poisson random variable with mean λ. For the next section, we will need a few more properties of the probability generating function, specifically on the interval $[0, 1]$. Assume that X is any nonnegative integer-valued random variable.

Exercise 8. *Show that* $f_X(s)$ *is continuous on the interval* $[0, 1]$. 1

Exercise 9. *Show that* $f_X(s)$ *is strictly increasing and that* $f'_X(s)$ *is non-decreasing.* 1

Conditional Expectation

Definition 4. *We define the* conditional expectation *of a random variable X, conditioned on an event A as*

$$\mathbb{E}(X \mid A) = \sum_x \mathbb{P}(X = x \mid A).$$

Exercise 10. *Show that for two random variables X and Y we have* 1

$$\mathbb{P}(X = k) = \sum_y \mathbb{P}(X = k \mid Y = y)\mathbb{P}(Y = y).$$

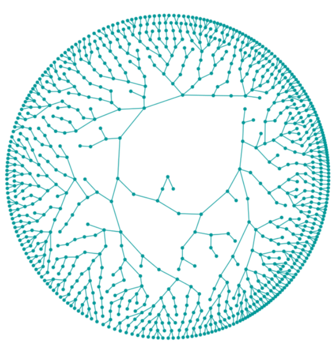
Exercise 11. *Show that for two random variables X and Y we have* 1

$$\mathbb{E}(X) = \sum_y \mathbb{E}(X \mid Y = y)\mathbb{P}(Y = y).$$

Branching Process

In this section we will introduce a branching process. This is a type of stochastic process that can be used to model reproductive or propagation processes, such as the growth of a population or the spread of disease on a network. In our case, we will use it as a model for an exploration of a connected component around a starting vertex in $G(n, p)$.

The *Galton-Watson* branching process is defined as follows. Fix a probability distribution $(p_0, p_1, p_2, \dots)$. In our case, we will use a Poisson distribution with mean λ, such that $p_k = \lambda^k e^{-\lambda}/k!$ for $k = 0, 1, 2, \dots$ Let Z be a random variable with this distribution. The process is defined as follows. At level (generation) 0, there is one root vertex. Each level $t+1$ consists of the children of vertices at level t. Each vertex has a number of children that is distributed as Z, but that is independent of all other vertices (at any level). You can see an example of a simulation here: `https://youtu.be/na0tu9aK820`.



Our aim is to show that when $\lambda \leq 1$, the process eventually dies out with probability 1, and when $\lambda > 1$, the process has a positive probability of survival. Let X_t be the number of individuals at level t. Note that X_t does not have a Poisson distribution, with the exception of $X_1 \sim Z$. We have that X_t is a sum of X_{t-1} Poisson random variables, with X_{t-1} being a random variable itself. If we condition on a fixed value of X_{t-1}, we find $\mathbb{E}(X_t)$ by using linearity of expectation:

$$\mathbb{E}(X_t \mid X_{t-1} = k) = k\lambda.$$

As in Exercise 11, this gives us

$$\mathbb{E}(X_t) = \sum_{k=0}^{\infty} \mathbb{E}(X_t \mid X_{t-1} = k)\mathbb{P}(X_{t-1} = k) = \sum_{k=0}^{\infty} k\lambda\mathbb{P}(X_{t-1} = k) = \lambda\mathbb{E}(X_{t-1}).$$

We let $\eta_t = \mathbb{P}(X_t = 0)$, and we let $\eta = \mathbb{P}(X_t = 0, \text{ for some } t)$. In other words, η_t is the probability that the process has died out by time t, and η is the probability that the process dies out at some point.

Exercise 12. *Show that when $\lambda < 1$, we have that $\eta = 1$.* 1

How does η_t depend on η_{t-1}? If $X_1 = k$, then we can view the process as a collection of k processes starting at time 1. The original process going extinct by time t is then equivalent to all k of these processes going extinct by time $t - 1$. This gives us that

$$\mathbb{P}(X_t = 0 \mid X_1 = k) = \eta_{t-1}^k.$$

We use this to find that

$$\eta_t = \mathbb{P}(X_t = 0) = \sum_{k=0}^{\infty} \mathbb{P}(X_t = 0 \mid X_1 = k)\mathbb{P}(X_1 = k) = \sum_{k=0}^{\infty} \eta_{t-1}^k \mathbb{P}(X_1 = k) = f_{X_1}(\eta_{t-1}),$$

where we note that $X_1 \sim Z$ has a Poisson distribution with mean λ.