Application of Harris-FKG

We look at one more application of Harris-FKG. In this example, it is not immediately clear how to apply this lemma, but it leads to a short and elegant proof of a (tight) upper bound on the size of certain set systems.

Theorem 1 (Daykin & Lovázs)**.** *Let $\mathcal{F}$ be a family of subsets of $[n]$, with the following two properties. For every $A, B \in \mathcal{F}$, we have*

(i) $A \cap B \neq \emptyset$,

(ii) $A \cup B \neq [n]$.

Then

$$|\mathcal{F}| \leq 2^{n-2}.$$

Proof. We will start by defining $\mathcal{F}$ as a set of events in a probability space. For any $A \in \mathcal{F}$, write A as a binary string of length n (with $0/1$ at index i indicating that i is excluded/included in A). Let $\Omega = \{0,1\}^n$ and sample uniformly from Ω. Then A is the event that the associated string is sampled, and $\mathcal{F}$ is the event that any set in $\mathcal{F}$ is sampled.

We define

$$\mathcal{G} = \{A' \mid A \subseteq A',\ A \in \mathcal{F}\},$$

and

$$\mathcal{H} = \{A' \mid A' \subseteq A,\ A \in \mathcal{F}\}.$$

Exercise 1. *Show that* 2

$$\mathcal{F} = \mathcal{G} \cap \mathcal{H}.$$

Exercise 2. *Show that* 2

$$\mathbb{P}(\mathcal{G}) \leq \frac{1}{2},\ \ \mathbb{P}(\mathcal{H}) \leq \frac{1}{2}.$$

Exercise 3. *Show that $\mathcal{G}$ is an increasing event on Ω, and $\mathcal{H}$ a decreasing event.* 2

By Harris' inequality, we have

$$\mathbb{P}(\mathcal{F}) = \mathbb{P}(\mathcal{G} \cap \mathcal{H}) \leq \mathbb{P}(\mathcal{H})\mathbb{P}(\mathcal{H}) = \frac{1}{4}.$$

Since we sample uniformly from Ω and $|\Omega| = 2^n$, we have

$$|\mathcal{F}| \leq 2^{n-2}.$$

□

Theorem 2 (Bloom)**.** *The bound in Theorem 1 is tight.*

Proof. We can construct a set system that satisfies the conditions in Theorem 1 as follows: Let

$$\mathcal{F} = \{A \subset [n] \mid 1 \in A,\ n \notin A\}.$$

Then it is easy to check that $\mathcal{F}$ satisfies the conditions. Furthermore, $\mathcal{F}$ is in bijection with the set of subsets of $[2, n-1]$, and therefore $|\mathcal{F}| = 2^{n-2}$. □

Ramsey numbers

We revisit the diagonal Ramsey numbers. Recall that $R(s,s)$ is defined as the least n such that every 2-coloring of the edges of K_n gives rise to a monochromatic K_s. We found a lower bound on $R(s,s)$ as one of our first exercises related to the first moment method. We revisited it again using the local lemma (although we didn't do the diagonal version then). Below are three lower bounds on $R(s,s)$ each time improving by a factor of $\sqrt{2}$. You should be able to prove each of them.

Theorem 3 (Erdös)**.** *If*

$$\binom{n}{k}2^{1-\binom{k}{2}} < 1,$$

then $R(s,s) > n$.

Obtaining an explicit bound is a bit challenging (here and in the following theorems), but it follows that

$$R(s,s) > \left(\frac{1}{e\sqrt{2}} + o(1)\right) k2^{k/2}.$$

Exercise 4. *Prove Theorem 3.* 2

Theorem 4. *For any n, s, we have*

$$R(s,s) > n - \binom{n}{k}2^{1-\binom{k}{2}}.$$

It follows that

$$R(s,s) > \left(\frac{1}{e} + o(1)\right) k2^{k/2}.$$

Exercise 5. *Prove Theorem 4. Hint: take a random 2-edge-coloring of K_n, and "get rid of" monochromatic copies of K_s.* 3

Theorem 5 (Spencer)**.** *If*

$$e2^{1-\binom{k}{2}}\left(\binom{k}{2}\binom{n}{k-2} + 1\right) < 1,$$

then then $R(s,s) > n$.

It follows that

$$R(s,s) > \left(\frac{\sqrt{2}}{e} + o(1)\right) k2^{k/2}.$$

Exercise 6. *Prove Theorem 5.* 3