Applications of Lovász Local Lemma

This week, we will look at a couple of applications of the local lemma. We will see that it is not always obvious that the local lemma applies to a situation, and even if we know that it does, how we should define the bad events and the dependency graph.

Independent sets

Exercise 1. *Show that every graph G with maximum degree $\Delta = \Delta(G)$ has an independent set of size at least $|V(G)|/(\Delta + 1)$.* 1

We will use the local lemma to prove something a little stronger about finding independent sets with a given structure.

Theorem 1. *Let G be a graph and let $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ be a partition of its vertices such that $|V_i| \geq 2e\Delta$ for $1 \leq i \leq k$. (Assume that $|V(G)| \geq 2e\Delta$.) Then G has an independent set with at least one vertex in every V_i.*

Proof. We set the proof up as follows. First, we assume that $|V_i| = \lceil 2e\Delta \rceil$ for all $1 \leq i \leq k$. Note that we can delete vertices from G if needed, and this will not affect the result.
Next, we choose a set of vertices S by choosing one vertex s_i from each V_i uniformly at random, and independently from other sets V_j. We then use the symmetric (non-lopsided) local lemma to show that with some positive probability S is an independent set.
Even with the given set-up, it is not immediately clear what the bad events should be. We have that S is an independent set if it contains no edges. So, in class we tried to call $A_{i,j}$ a bad event when the vertex s_i chosen from the set V_i and the vertex s_j chosen from the set V_j share an edge. In this case, we can bound $\mathbb{P}(A_{i,j}) \leq \Delta/\lceil 2e\Delta \rceil \leq 1/(2e)$. The argument here is that we can choose s_i first, and then it has at most Δ neighbors in the set V_j. However, we quickly run into trouble finding a suitable dependency graph and value d that would satisfy the condition of the local lemma. (In fact, we would need $d = 1$, which is impossible.) In any case, roughly speaking, our d in the dependency graph is likely going to be a multiple of Δ, so we need p to be a multiple of $1/\Delta$.
Instead, we try the following. Let f be an edge in G, and let A_f be the event that it (i.e. both of its endpoints) is chosen in S. If no bad events happen, then S is an independent set. Although this set-up sounds very similar to the previous attempt, it has an advantage, since it gives us

$$\mathbb{P}(A_f) \leq \frac{1}{\lceil 2e\Delta \rceil^2}.$$

Either both of the endpoints of f lie in the same set V_i, in which case $\mathbb{P}(A_f) = 0$, and otherwise the above probability holds, since we need to choose both of its endpoints for S, and they are chosen independently.

Exercise 2. *Finish this proof.* 2

□

Cycles of length divisible by k

For simple graphs, is there a minimum degree δ that guarantees the existence of even cycles? Note that this does not work for odd cycles: complete bipartite graphs may have arbitrarilty high degree without having odd cycles. Surprisingly, we are able to guarantee cycles of length divisible by k for any $k \geq 1$. We start with a result on directed graphs.

Theorem 2 (Alon & Linial)**.** *Every directed graph with minimum out-degree δ and maximum in-degree Δ has a directed cycle of length divisible by k if*

$$k \leq \frac{\delta}{1 + \log(1 + \delta\Delta)}.$$

Proof. First, assume that every vertex has out-degree exactly δ. We can achieve this by deleting edges of G if necessary. We use the following set-up. Let $[k] = \{0, \ldots, k-1\}$. We will find a random coloring $c : V(G) \to [k]$ of the vertices of G. We color each vertex with a color chosen uniformly from $[k]$ independently of other vertices, as we have done before with 2-colorings. Then, we keep edges $v \to u$ if $c(u) \equiv c(v) + 1 \pmod{k}$. If, in the remaining graph, every vertex has out-degree at least 1, then we are guaranteed to have a directed cycle of length divisible by k. (Why?) For each $v \in V(G)$, we let A_v be the event that v does not have an out-going edge. For each neighbor u of v, the probability that the edge $v \to u$ survives is $1 - 1/k$. Since the neighbors are colored independently, we have

$$\mathbb{P}(A_v) = (1 - 1/k)^\delta \leq e^{-\delta/k},$$

for all $v \in V(G)$.

Exercise 3. *Finish this proof.* 4

□

Exercise 4. *Prove, as a corollary, that for every k there exists a d such that every $2d$-regular undirected graph has a cycle of length divisible by k.* 2

Latin transversals

A Latin square is an $n \times n$ array of values in $[n]$ such no value appears twice in the same row or column. A transversal is a set of n entries such that no two appear in the same row or column. A Latin transversal is a transversal with distinct entries. It is a famous open conjecture that every Latin square of odd order has a Latin transversal. (It is known that even order Latin squares need not have one.) A partial result towards this conjecture was the original application of the lopsided local lemma.

Theorem 3 (Erdős & Spencer)**.** *Every $n \times n$ array in which every value appears at most $n/(4e)$ times has a Latin transversal.*

Proof. Suppose that the array is given. We cannot randomize the values in the array, so we will randomize the chosen transversal, by choosing one uniformly from all possible transversals. Note that a transversal is equivalent to a permutation, or a bijection between rows and columns. So, there are $n!$ possible transversals. For every pair of entries $(i,j), (k,l)$ that have equal values as their entries and that do not share a row or column, let A_{ijkl} be the bad event that both are chosen in our random transversal.

Exercise 5. *First, show that* 1

$$\mathbb{P}(A_{ijkl}) = \frac{1}{n(n-1)}.$$

For the (lopsided) dependency graph, we should add edges from A_{ijkl} to A_{qrst} whenever they share a row or column. In other words, whenever the set of entries $(i, j), (k, l), (q, r), (s, t)$ are not a subset of any transversal.

Exercise 6. *Show that the maximum degree in this dependency graph is at most* 4

$$(4n - 4)(\tfrac{n}{4e} - 1),$$

and finish the proof.

$\square$